16.001 - Materials & Structures
Fall 2020
Problem Set #7

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Assigned:	Thursday, Oct. 15th, 5:00 PM
Due:	Thursday, Oct. 22nd, 3:00 PM

Question	Points
1	0
2	0
3	0
4	9
5	7
Total:	16

○ **Problems M-7.1** [0 points]
(M.O. M8)

1.1 Suppose a body in equilibrium (assume no body forces in this problem) experiences a stress field in which the normal stress in the $\mathbf{e}_1$ direction is a function of x_2, the normal stress in the $\mathbf{e}_2$ direction is a function of x_1, and all out of plane stress components are zero, so that the stress field has the following form:

$$\sigma_{11} = f(x_2)$$
$$\sigma_{22} = g(x_1)$$
$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

Show that the in-plane shear stress σ_{12} must be a constant value.

Solution: The equilibrium equations are (assuming no body forces and using the fact that the stress tensor is symmetric):

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{13}}{\partial x_3}} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{23}}{\partial x_3}} = 0$$
$$\cancelto{0}{\frac{\partial \sigma_{31}}{\partial x_1}} + \cancelto{0}{\frac{\partial \sigma_{32}}{\partial x_2}} + \cancelto{0}{\frac{\partial \sigma_{33}}{\partial x_3}} = 0$$

and simplify to:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Substituting for σ_{11} and σ_{22}:

$$\cancelto{0}{\frac{\partial f(x_2)}{\partial x_1}} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \cancelto{0}{\frac{\partial g(x_1)}{\partial x_2}} = 0$$

So we find that

$$\frac{\partial \sigma_{12}}{\partial x_1} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_2} = 0$$

where $\sigma_{12} = \sigma_{21}$ by symmetry of the stress tensor. This is only possible if $\boxed{\sigma_{12} = \text{constant}}$.

1.2 Consider the following stress field for the body

$$\sigma_{11} = x_2^2 + x_2 + 1$$
$$\sigma_{22} = x_1^2 + x_1 + 1$$
$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

Give a possible value for σ_{12} so that the body is in equilibrium. Justify your answer.

Solution: The stress field above is simply a specific example of the stress field given in part (a), where $\sigma_{11} = f(x_2) = x_2^2 + x_2 + 1$ and $\sigma_{22} = g(x_1) = x_1^2 + x_1 + 1$. We know that for such a stress field, any $\boxed{\sigma_{12} = \text{constant}}$ is valid. One possible (and the simplest) value for σ_{12} is 0.

1.3 Now consider the following stress field ($\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$)

$$\sigma_{11} = x_1^2 + x_2^2$$
$$\sigma_{22} = x_1^2 + x_2^2$$
$$\sigma_{12} = -2x_1x_2$$

Determine if the body is in equilibrium.

Solution: The equilibrium equations for this case are:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Substituting our stress components σ_{11}, σ_{12} and σ_{22}:

$$\frac{\partial (x_1^2 + x_2^2)}{\partial x_1} + \frac{\partial (-2x_1x_2)}{\partial x_2} = (2x_1) + (-2x_1) = 0$$
$$\frac{\partial (-2x_1x_2)}{\partial x_1} + \frac{\partial (x_1^2 + x_2^2)}{\partial x_2} = (-2x_2) + (2x_2) = 0$$

Since the equilibrium equations are satisfied, <u>the body is in equilibrium</u>.

○ **Problems M-7.2** [0 points]
Stress fields in static equilibrium.

Let's consider a structure in equilibrium and free of body forces. Are the following stress fields possible?

2.1 $\boldsymbol{\sigma} = \begin{bmatrix} c_1x_1 + c_2x_2 + c_3x_1x_2 & -c_3\dfrac{x_2^2}{2} - c_1x_2 \\ -c_3\dfrac{x_2^2}{2} - c_1x_2 & c_4x_1 + c_1x_2 \end{bmatrix}.$

2.2 $\boldsymbol{\sigma} = \begin{bmatrix} 3x_1 + 5x_2 & 4x_1 - 3x_2 \\ 4x_1 - 3x_2 & 2x_1 - 4x_2 \end{bmatrix}.$

2.3 $\boldsymbol{\sigma} = \begin{bmatrix} x_1^2 - 2x_1x_2 + cx_3 & -x_1x_2 + x_2^2 & -x_1x_3 \\ -x_1x_2 + x_2^2 & x_2^2 & -x_2x_3 \\ -x_1x_3 & -x_2x_3 & (x_1+x_2)x_3 \end{bmatrix}.$

Solution: To solve this problem we turn to the momentum equation

$$\frac{\partial \sigma_{ji}}{\partial x_j} + \rho f_i = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

As the structure is in equilibrium (steady state) and free of body forces, the terms $\rho\frac{\partial^2 u_i}{\partial t^2}$ and ρf_i are null. Then, the equilibrium equations become

$$\frac{\partial \sigma_{ji}}{\partial x_j} = 0.$$

For a 2D stress field we have:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} \\ \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} \end{bmatrix} \tag{1}$$

For a 3D stress field we have:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} \dfrac{\partial \sigma_{11}}{\partial x_1} + \dfrac{\partial \sigma_{12}}{\partial x_2} + \dfrac{\partial \sigma_{13}}{\partial x_3} \\ \\ \dfrac{\partial \sigma_{21}}{\partial x_1} + \dfrac{\partial \sigma_{22}}{\partial x_2} + \dfrac{\partial \sigma_{23}}{\partial x_3} \\ \\ \dfrac{\partial \sigma_{31}}{\partial x_1} + \dfrac{\partial \sigma_{32}}{\partial x_2} + \dfrac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} \tag{2}$$

1. For the first stress field we obtained:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} c_1 + c_3x_2 + -c_3x_2 - c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \end{bmatrix} \tag{3}$$

This stress field does not satisfy the equilibrium equation.

2. For the second stress field we obtained:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} 3-3 \\ 4-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

This stress field does satisfy the equilibrium equation.

3. For the third stress field we obtained:

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \begin{bmatrix} 2x_1 - 2x_2 - x_1 + 2x_2 - x_1 \\ -x_2 + 2x_2 - x_2 \\ -x_3 - x_3 + x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2x_3 + x_1 + x_2 \end{bmatrix} \tag{5}$$

This stress field does not satisfy the equilibrium equation

○ **Problems M-7.3** [0 points]

For certain problems whose geometry features a rotational symmetry about one of its axes, Cartesian coordinates may not be the most convenient choice. For such problems, cylindrical coordinates may be more convenient for describing the involved field quantities.

The objective of this problem is to specialize the "stress equilibrium" equations in vector form

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{f} \;=\; \mathbf{0}, \tag{6}$$

which were introduced in class in terms of Cartesian coordinates as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \tag{7}$$

to cylindrical coordinates.

The end result we want to find for the full 3D case is:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0 \tag{8}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta = 0 \tag{9}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0 \tag{10}$$

We start by looking at the representation of a vector $\mathbf{r}$ in terms of both the Cartesian coordinates x_i and the cylindrical coordinates r, θ, z, Figure 1.

3.1 Express the Cartesian coordinates x_1, x_2, x_3 in terms of the cylindrical coordinates r, θ, z, and vice versa.

Solution:

$$x_1 = r\cos(\theta) \tag{11}$$

$$x_2 = r\sin(\theta) \tag{12}$$

$$x_3 = z \tag{13}$$

$$r = \sqrt{x_1^2 + x_2^2} \tag{14}$$

$$\theta = \arctan\left(\frac{x_2}{x_1}\right) \tag{15}$$

$$z = x_3 \tag{16}$$

3.2 Express the orthonormal basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ pertaining to the Cartesian coordinate system in terms of the orthonormal basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ pertaining to the cylindrical coordinate system (and any coordinates you may need).

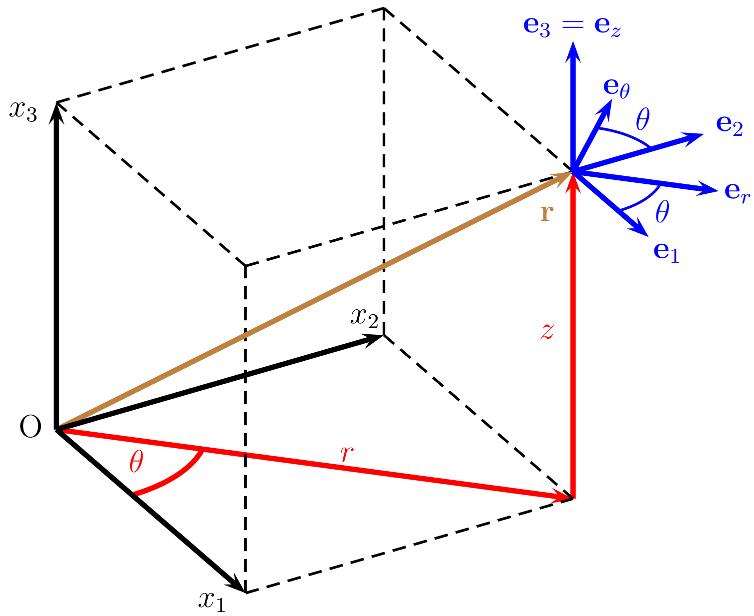


Figure 1: Position vector $\mathbf{r}$ represented in terms of the Cartesian coordinates x_i (black) and the cylindrical coordinates r, θ, z (red). The respective unit basis vectors pertaining to the position $\mathbf{r}$ are also displayed.

Solution:

Basis vectors of the Cartesian coordinate system in terms of the basis vectors of the cylindrical coordinate system:

$$\mathbf{e}_1 = \mathbf{e}_r \cos(\theta) - \mathbf{e}_\theta \sin(\theta) \quad (17)$$

$$\mathbf{e}_2 = \mathbf{e}_r \sin(\theta) + \mathbf{e}_\theta \cos(\theta) \quad (18)$$

$$\mathbf{e}_3 = \mathbf{e}_z \quad (19)$$

3.3 Using your results from the previous parts, show that the del operator $\boldsymbol{\nabla}$ which is given in Cartesian coordinates as

$$\boldsymbol{\nabla} = \mathbf{e}_1 \frac{\partial()}{\partial x_1} + \mathbf{e}_2 \frac{\partial()}{\partial x_2} + \mathbf{e}_3 \frac{\partial()}{\partial x_3} \quad (20)$$

can be expressed in cylindrical coordinates as

$$\boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial()}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial()}{\partial \theta} + \mathbf{e}_z \frac{\partial()}{\partial z}. \quad (21)$$

Suggested approach:

1. Insert $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ from Part (b) into Eq. (20).
2. Use the chain rule

$$\frac{\partial()}{\partial x_i} = \frac{\partial()}{\partial r}\frac{\partial r}{\partial x_i} + \frac{\partial()}{\partial \theta}\frac{\partial \theta}{\partial x_i} + \frac{\partial()}{\partial z}\frac{\partial z}{\partial x_i} \quad \text{for} \quad i \in \{1, 2, 3\} \quad (22)$$

together with your results from Part (a) to replace $\partial()/\partial x_1$, $\partial()/\partial x_2$, $\partial()/\partial x_3$ in Eq. (20).

Solution: Following the suggested approach, we can insert $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ from Part (b) into Eq. (20):

$$\boldsymbol{\nabla} = (\mathbf{e}_r\cos(\theta) - \mathbf{e}_\theta\sin(\theta))\frac{\partial()}{\partial x_1} + (\mathbf{e}_r\sin(\theta) + \mathbf{e}_\theta\cos(\theta))\frac{\partial()}{\partial x_2} + \mathbf{e}_z\frac{\partial()}{\partial x_3}$$

Next, we insert the chain rule stated in Eq. (22):

$$\begin{aligned}\boldsymbol{\nabla} = \quad & (\mathbf{e}_r\cos(\theta) - \mathbf{e}_\theta\sin(\theta))\left(\frac{\partial()}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial()}{\partial \theta}\frac{\partial \theta}{\partial x_1} + \frac{\partial()}{\partial z}\frac{\partial z}{\partial x_1}\right)\\ &+ (\mathbf{e}_r\sin(\theta) + \mathbf{e}_\theta\cos(\theta))\left(\frac{\partial()}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial()}{\partial \theta}\frac{\partial \theta}{\partial x_2} + \frac{\partial()}{\partial z}\frac{\partial z}{\partial x_2}\right)\\ &+ \mathbf{e}_z\left(\frac{\partial()}{\partial r}\frac{\partial r}{\partial x_3} + \frac{\partial()}{\partial \theta}\frac{\partial \theta}{\partial x_3} + \frac{\partial()}{\partial z}\frac{\partial z}{\partial x_3}\right)\end{aligned}$$

In order to simplify the above expression further, we need to evaluate the partial derivatives occurring in it:

$$\begin{aligned}
\frac{\partial r}{\partial x_1} &= \frac{x_1}{\sqrt{x_1^2+x_2^2}} = \cos(\theta) & \frac{\partial r}{\partial x_2} &= \frac{x_2}{\sqrt{x_1^2+x_2^2}} = \sin(\theta) & \frac{\partial r}{\partial x_3} &= 0\\
\frac{\partial \theta}{\partial x_1} &= \frac{-x_2}{x_1^2+x_2^2} = -\frac{\sin(\theta)}{r} & \frac{\partial \theta}{\partial x_2} &= \frac{x_1}{x_1^2+x_2^2} = \frac{\cos(\theta)}{r} & \frac{\partial \theta}{\partial x_3} &= 0\\
\frac{\partial z}{\partial x_1} &= 0 & \frac{\partial z}{\partial x_2} &= 0 & \frac{\partial z}{\partial x_3} &= 1
\end{aligned}$$

Inserting the partial derivatives yields

$$\begin{aligned}\boldsymbol{\nabla} = \quad & (\mathbf{e}_r\cos(\theta) - \mathbf{e}_\theta\sin(\theta))\left(\frac{\partial()}{\partial r}\cos(\theta) - \frac{\partial()}{\partial \theta}\frac{1}{r}\sin(\theta)\right)\\ &+ (\mathbf{e}_r\sin(\theta) + \mathbf{e}_\theta\cos(\theta))\left(\frac{\partial()}{\partial r}\sin(\theta) + \frac{\partial()}{\partial \theta}\frac{1}{r}\cos(\theta)\right)\\ &+ \mathbf{e}_z\frac{\partial()}{\partial z}\end{aligned}$$

which can be rearranged as

$$\begin{aligned}\boldsymbol{\nabla} = \quad & \mathbf{e}_r\left(\frac{\partial()}{\partial r}\cos^2(\theta) + \frac{\partial()}{\partial r}\sin^2(\theta) + \frac{\partial()}{\partial \theta}\frac{1}{r}\sin(\theta)\cos(\theta) - \frac{\partial()}{\partial \theta}\frac{1}{r}\sin(\theta)\cos(\theta)\right)\\ &+ \mathbf{e}_\theta\left(\frac{\partial()}{\partial \theta}\frac{1}{r}\cos^2(\theta) + \frac{\partial()}{\partial \theta}\frac{1}{r}\sin^2(\theta) + \frac{\partial()}{\partial r}\sin(\theta)\cos(\theta) - \frac{\partial()}{\partial r}\sin(\theta)\cos(\theta)\right)\\ &+ \mathbf{e}_z\frac{\partial()}{\partial z}\\ = \quad & \mathbf{e}_r\frac{\partial()}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial()}{\partial \theta} + \mathbf{e}_z\frac{\partial()}{\partial z}.\end{aligned}$$

3.4 Express the stress tensor $\boldsymbol{\sigma}$ in terms of the unit basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ and its respective stress components.

Solution:

$$\begin{aligned}\boldsymbol{\sigma} \;=\; & \sigma_{rr}\,\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{r\theta}\,\mathbf{e}_r \otimes \mathbf{e}_\theta + \sigma_{rz}\,\mathbf{e}_r \otimes \mathbf{e}_z \\ & + \sigma_{\theta r}\,\mathbf{e}_\theta \otimes \mathbf{e}_r + \sigma_{\theta\theta}\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\theta z}\,\mathbf{e}_\theta \otimes \mathbf{e}_z \\ & + \sigma_{zr}\,\mathbf{e}_z \otimes \mathbf{e}_r + \sigma_{z\theta}\,\mathbf{e}_z \otimes \mathbf{e}_\theta + \sigma_{zz}\,\mathbf{e}_z \otimes \mathbf{e}_z \end{aligned} \tag{23}$$

3.5 Express the divergence of the stress tensor $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}$ in terms of the unit basis vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ and the respective stress components. Do this by combining Eq. (21) and your representation of the stress tensor from the previous part. Recall that the stress tensor is symmetric and that

$$\mathbf{e}_i \cdot (\mathbf{e}_j \otimes \mathbf{e}_k) \;=\; (\mathbf{e}_i \cdot \mathbf{e}_j)\,\mathbf{e}_k \;=\; \delta_{ij}\,\mathbf{e}_k \quad \text{for} \quad i,j,k \in \{r,\theta,z\}\,. \tag{24}$$

Solution:
The formal application of the del operator in cylindrical coordinates as stated in Eq. (21) yields:

$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \;=\; & \left(\mathbf{e}_r \frac{\partial()}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial()}{\partial \theta} + \mathbf{e}_z \frac{\partial()}{\partial z}\right) \cdot \boldsymbol{\sigma} \\
=\; & \quad \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{rr}\,\mathbf{e}_r \otimes \mathbf{e}_r) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{rr}\,\mathbf{e}_r \otimes \mathbf{e}_r) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{rr}\,\mathbf{e}_r \otimes \mathbf{e}_r) \\
& + \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{r\theta}\,\mathbf{e}_r \otimes \mathbf{e}_\theta) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{r\theta}\,\mathbf{e}_r \otimes \mathbf{e}_\theta) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{r\theta}\,\mathbf{e}_r \otimes \mathbf{e}_\theta) \\
& + \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{rz}\,\mathbf{e}_r \otimes \mathbf{e}_z) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{rz}\,\mathbf{e}_r \otimes \mathbf{e}_z) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{rz}\,\mathbf{e}_r \otimes \mathbf{e}_z) \\
& + \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{\theta r}\,\mathbf{e}_\theta \otimes \mathbf{e}_r) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta r}\,\mathbf{e}_\theta \otimes \mathbf{e}_r) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{\theta r}\,\mathbf{e}_\theta \otimes \mathbf{e}_r) \\
& + \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{\theta\theta}\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta\theta}\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{\theta\theta}\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta) \\
& + \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{\theta z}\,\mathbf{e}_\theta \otimes \mathbf{e}_z) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta z}\,\mathbf{e}_\theta \otimes \mathbf{e}_z) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{\theta z}\,\mathbf{e}_\theta \otimes \mathbf{e}_z) \\
& + \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{zr}\,\mathbf{e}_z \otimes \mathbf{e}_r) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{zr}\,\mathbf{e}_z \otimes \mathbf{e}_r) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{zr}\,\mathbf{e}_z \otimes \mathbf{e}_r) \\
& + \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{z\theta}\,\mathbf{e}_z \otimes \mathbf{e}_\theta) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{z\theta}\,\mathbf{e}_z \otimes \mathbf{e}_\theta) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{z\theta}\,\mathbf{e}_z \otimes \mathbf{e}_\theta) \\
& + \mathbf{e}_r \cdot \frac{\partial}{\partial r}(\sigma_{zz}\,\mathbf{e}_z \otimes \mathbf{e}_z) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{zz}\,\mathbf{e}_z \otimes \mathbf{e}_z) + \mathbf{e}_z \cdot \frac{\partial}{\partial z}(\sigma_{zz}\,\mathbf{e}_z \otimes \mathbf{e}_z)
\end{aligned}$$

Before further evaluating the above expression, an important difference between Cartesian and cylindrical coordinates shall be pointed out: While the directions of the basis vectors are constant in a Cartesian coordinate system, the directions of the basis vectors in a cylindrical coordinate system depend on (the coordinates of) the considered point. This can be seen by expressing the basis vectors of the cylindrical coordinate system in terms of the (spatially constant) basis vectors $\mathbf{e}_i$ of a Cartesian coordinate system:

$$\mathbf{e}_r = \mathbf{e}_1 \cos(\theta) + \mathbf{e}_2 \sin(\theta) \tag{25}$$
$$\mathbf{e}_\theta = -\mathbf{e}_1 \sin(\theta) + \mathbf{e}_2 \cos(\theta) \tag{26}$$
$$\mathbf{e}_z = \mathbf{e}_3 \tag{27}$$

Obviously, both $\mathbf{e}_r$ and $\mathbf{e}_\theta$ depend on θ. Consequently, their respective spatial derivatives are non-zero

$$\frac{\partial}{\partial \theta}(\mathbf{e}_r) = \frac{\partial}{\partial \theta}(\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2) = -\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2 = \mathbf{e}_\theta \tag{28}$$
$$\frac{\partial}{\partial \theta}(\mathbf{e}_\theta) = \frac{\partial}{\partial \theta}(-\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2) = -\cos(\theta)\mathbf{e}_1 - \sin(\theta)\mathbf{e}_2 = -\mathbf{e}_r \tag{29}$$

while all other spatial derivatives of the basis vectors of a cylindrical coordinate system are the zero vector:

$$\frac{\partial}{\partial r}(\mathbf{e}_r) = \frac{\partial}{\partial r}(\mathbf{e}_\theta) = \frac{\partial}{\partial r}(\mathbf{e}_z) = \frac{\partial}{\partial z}(\mathbf{e}_r) = \frac{\partial}{\partial z}(\mathbf{e}_\theta) = \frac{\partial}{\partial z}(\mathbf{e}_z) = \frac{\partial}{\partial \theta}(\mathbf{e}_z) = \mathbf{0}.$$

Taking this insight into account, the application of the product rule yields:

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \; & \mathbf{e}_r \cdot \frac{\partial \sigma_{rr}}{\partial r} (\mathbf{e}_r \otimes \mathbf{e}_r) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{rr}}{\partial \theta} (\mathbf{e}_r \otimes \mathbf{e}_r) + \mathbf{e}_z \cdot \frac{\partial \sigma_{rr}}{\partial z} (\mathbf{e}_r \otimes \mathbf{e}_r) \\
& + \mathbf{e}_r \cdot \frac{\partial \sigma_{r\theta}}{\partial r} (\mathbf{e}_r \otimes \mathbf{e}_\theta) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} (\mathbf{e}_r \otimes \mathbf{e}_\theta) + \mathbf{e}_z \cdot \frac{\partial \sigma_{r\theta}}{\partial z} (\mathbf{e}_r \otimes \mathbf{e}_\theta) \\
& + \mathbf{e}_r \cdot \frac{\partial \sigma_{rz}}{\partial r} (\mathbf{e}_r \otimes \mathbf{e}_z) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{rz}}{\partial \theta} (\mathbf{e}_r \otimes \mathbf{e}_z) + \mathbf{e}_z \cdot \frac{\partial \sigma_{rz}}{\partial z} (\mathbf{e}_r \otimes \mathbf{e}_z) \\
& + \mathbf{e}_r \cdot \frac{\partial \sigma_{\theta r}}{\partial r} (\mathbf{e}_\theta \otimes \mathbf{e}_r) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} (\mathbf{e}_\theta \otimes \mathbf{e}_r) + \mathbf{e}_z \cdot \frac{\partial \sigma_{\theta r}}{\partial z} (\mathbf{e}_\theta \otimes \mathbf{e}_r) \\
& + \mathbf{e}_r \cdot \frac{\partial \sigma_{\theta\theta}}{\partial r} (\mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} (\mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \mathbf{e}_z \cdot \frac{\partial \sigma_{\theta\theta}}{\partial z} (\mathbf{e}_\theta \otimes \mathbf{e}_\theta) \\
& + \mathbf{e}_r \cdot \frac{\partial \sigma_{\theta z}}{\partial r} (\mathbf{e}_\theta \otimes \mathbf{e}_z) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} (\mathbf{e}_\theta \otimes \mathbf{e}_z) + \mathbf{e}_z \cdot \frac{\partial \sigma_{\theta z}}{\partial z} (\mathbf{e}_\theta \otimes \mathbf{e}_z) \\
& + \mathbf{e}_r \cdot \frac{\partial \sigma_{zr}}{\partial r} (\mathbf{e}_z \otimes \mathbf{e}_r) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{zr}}{\partial \theta} (\mathbf{e}_z \otimes \mathbf{e}_r) + \mathbf{e}_z \cdot \frac{\partial \sigma_{zr}}{\partial z} (\mathbf{e}_z \otimes \mathbf{e}_r) \\
& + \mathbf{e}_r \cdot \frac{\partial \sigma_{z\theta}}{\partial r} (\mathbf{e}_z \otimes \mathbf{e}_\theta) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} (\mathbf{e}_z \otimes \mathbf{e}_\theta) + \mathbf{e}_z \cdot \frac{\partial \sigma_{z\theta}}{\partial z} (\mathbf{e}_z \otimes \mathbf{e}_\theta) \\
& + \mathbf{e}_r \cdot \frac{\partial \sigma_{zz}}{\partial r} (\mathbf{e}_z \otimes \mathbf{e}_z) + \mathbf{e}_\theta \cdot \frac{1}{r}\frac{\partial \sigma_{zz}}{\partial z} (\mathbf{e}_z \otimes \mathbf{e}_z) + \mathbf{e}_z \cdot \frac{\partial \sigma_{zz}}{\partial z} (\mathbf{e}_z \otimes \mathbf{e}_z) \\
& + \frac{\sigma_{rr}}{r} \mathbf{e}_\theta \cdot \Bigg[\underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_r)}_{=\mathbf{e}_\theta} \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_r)}_{=\mathbf{e}_\theta} \Bigg] \\
& + \frac{\sigma_{r\theta}}{r} \mathbf{e}_\theta \cdot \Bigg[\underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_r)}_{=\mathbf{e}_\theta} \otimes \mathbf{e}_\theta + \mathbf{e}_r \otimes \underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_\theta)}_{=-\mathbf{e}_r} \Bigg] \\
& + \frac{\sigma_{rz}}{r} \mathbf{e}_\theta \cdot \Bigg[\underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_r)}_{=\mathbf{e}_\theta} \otimes \mathbf{e}_z \Bigg] \\
& + \frac{\sigma_{\theta r}}{r} \mathbf{e}_\theta \cdot \Bigg[\underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_\theta)}_{=-\mathbf{e}_r} \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_r)}_{=\mathbf{e}_\theta} \Bigg] \\
& + \frac{\sigma_{\theta\theta}}{r} \mathbf{e}_\theta \cdot \Bigg[\underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_\theta)}_{=-\mathbf{e}_r} \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_\theta)}_{=-\mathbf{e}_r} \Bigg] \\
& + \frac{\sigma_{\theta z}}{r} \mathbf{e}_\theta \cdot \Bigg[\underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_\theta)}_{=-\mathbf{e}_r} \otimes \mathbf{e}_z \Bigg] \\
& + \frac{\sigma_{zr}}{r} \mathbf{e}_\theta \cdot \Bigg[\mathbf{e}_z \otimes \underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_r)}_{=\mathbf{e}_\theta} \Bigg] + \frac{\sigma_{z\theta}}{r} \mathbf{e}_\theta \cdot \Bigg[\mathbf{e}_z \otimes \underbrace{\frac{\partial}{\partial \theta}(\mathbf{e}_\theta)}_{=-\mathbf{e}_r} \Bigg]
\end{aligned} \tag{30}
$$

Eq. (30) can be simplified using the hint given in Eq. (24):

$$\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} \;=\;& \frac{\partial \sigma_{rr}}{\partial r}\mathbf{e}_r + \frac{\partial \sigma_{r\theta}}{\partial r}\mathbf{e}_\theta + \frac{\partial \sigma_{rz}}{\partial r}\mathbf{e}_z \\
&+ \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta}\mathbf{e}_r + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}\mathbf{e}_z \\
&+ \frac{\partial \sigma_{zr}}{\partial z}\mathbf{e}_r + \frac{\partial \sigma_{z\theta}}{\partial z}\mathbf{e}_\theta + \frac{\partial \sigma_{zz}}{\partial z}\mathbf{e}_z \\
&+ \frac{\sigma_{rr}}{r}\,\mathbf{e}_r + \frac{\sigma_{r\theta}}{r}\,\mathbf{e}_\theta + \frac{\sigma_{rz}}{r}\,\mathbf{e}_z \\
&+ \frac{\sigma_{\theta r}}{r}\,\mathbf{e}_\theta - \frac{\sigma_{\theta\theta}}{r}\,\mathbf{e}_r \\
=\;& \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr}}{r} - \frac{\sigma_{\theta\theta}}{r}\right)\mathbf{e}_r \\
&+ \left(\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\sigma_{r\theta}}{r} + \frac{\sigma_{\theta r}}{r}\right)\mathbf{e}_\theta \\
&+ \left(\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}\right)\mathbf{e}_z
\end{aligned} \tag{31}$$

Finally, exploiting the symmetry of the stress tensor $\boldsymbol{\sigma}$, one finds:

$$\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} \;=\;& \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right)\mathbf{e}_r \\
&+ \left(\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r}\right)\mathbf{e}_\theta \\
&+ \left(\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}\right)\mathbf{e}_z
\end{aligned} \tag{32}$$

3.6 Express the stress equilibrium in Eq. (6) in terms of cylindrical coordinates and the associated unit basis vectors.

Solution:

$$\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} \;=\;& \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}\right)\mathbf{e}_r \\
&+ \left(\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r}\right)\mathbf{e}_\theta \\
&+ \left(\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}\right)\mathbf{e}_z \\
&+ f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z \\
=\;& \mathbf{0}
\end{aligned} \tag{33}$$

○ **Problems M-7.4** [9 points]
(M.O. M8)

4.1 (3 points) Consider a material element under plane stress, so that all out of plane stress components $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$. Body forces f_1 and f_2 act on the material element as well. Derive the equilibrium equations in the case of plane stress.

Solution: We derive the equations of equilibrium corresponding to plane stress by applying force equilibrium in the 1 and 2 directions. Note that each stress must be multiplied by a corresponding area to obtain force, which in this 2D case is essentially a length.

- Force Equilibrium in 1-direction

$$\sum F_1 = 0 =$$
$$\left(\cancel{\sigma_{11}} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) dx_2 - \cancel{\sigma_{11}} dx_2$$
$$+ \left(\cancel{\sigma_{21}} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) dx_1 - \cancel{\sigma_{21}} dx_1 + f_1$$
$$\boxed{\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + f_1 = 0}$$

- Force Equilibrium in 2-direction

$$\sum F_2 = 0 =$$
$$\left(\cancel{\sigma_{22}} + \frac{\partial \sigma_{22}}{\partial x_2} dx_2\right) dx_1 - \cancel{\sigma_{22}} dx_1$$
$$+ \left(\cancel{\sigma_{12}} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1\right) dx_2 - \cancel{\sigma_{12}} dx_2 + f_2$$
$$\boxed{\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0}$$

Thus, the two equations of equilibrium corresponding to the case of plane stress are:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + f_1 = 0 \qquad (34)$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0 \qquad (35)$$

4.2 (3 points) Suppose a body in equilibrium (assume no body forces in this problem) experiences a stress field in which the normal stress in the $\mathbf{e}_1$ direction is a function of x_2, the normal stress in the $\mathbf{e}_2$ direction is a function of x_1, and all out of plane stress components are zero, so that the stress field has the following form:

$$\sigma_{11} = f(x_2)$$
$$\sigma_{22} = g(x_1)$$
$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

Show that the in-plane shear stress σ_{12} must be a constant value.

Solution: The equilibrium equations are (assuming no body forces and using the fact that the stress tensor is symmetric):

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{13}}{\partial x_3}} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \cancelto{0}{\frac{\partial \sigma_{23}}{\partial x_3}} = 0$$
$$\cancelto{0}{\frac{\partial \sigma_{31}}{\partial x_1}} + \cancelto{0}{\frac{\partial \sigma_{32}}{\partial x_2}} + \cancelto{0}{\frac{\partial \sigma_{33}}{\partial x_3}} = 0$$

and simplify to:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Substituting for σ_{11} and σ_{22}:

$$\cancelto{0}{\frac{\partial f(x_2)}{\partial x_1}} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \cancelto{0}{\frac{\partial g(x_1)}{\partial x_2}} = 0$$

So we find that

$$\frac{\partial \sigma_{12}}{\partial x_1} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_2} = 0$$

where $\sigma_{12} = \sigma_{21}$ by symmetry of the stress tensor. This is only possible if $\boxed{\sigma_{12} = \text{constant}}$.

4.3 (1 point) Consider the following stress field for the body

$$\sigma_{11} = x_2^2 + x_2 + 1$$
$$\sigma_{22} = x_1^2 + x_1 + 1$$
$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

Give a possible value for σ_{12} so that the body is in equilibrium. Justify your answer.

Solution: The stress field above is simply a specific example of the stress field given in part (a), where $\sigma_{11} = f(x_2) = x_2^2 + x_2 + 1$ and $\sigma_{22} = g(x_1) = x_1^2 + x_1 + 1$. We know that for such a stress field, any $\boxed{\sigma_{12} = \text{constant}}$ is valid. One possible (and the simplest) value for σ_{12} is 0.

4.4 (2 points) Now consider the following stress field ($\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$)

$$\sigma_{11} = x_1^2 + x_2^2$$
$$\sigma_{22} = x_1^2 + x_2^2$$
$$\sigma_{12} = -2x_1x_2$$

Determine if there can be any body forces for the body to be in equilibrium.

Solution: The equilibrium equations for this case are:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0$$

Substituting our stress components σ_{11}, σ_{12} and σ_{22}:

$$\frac{\partial(x_1^2 + x_2^2)}{\partial x_1} + \frac{\partial(-2x_1x_2)}{\partial x_2} = (2x_1) + (-2x_1) = 0$$
$$\frac{\partial(-2x_1x_2)}{\partial x_1} + \frac{\partial(x_1^2 + x_2^2)}{\partial x_2} = (-2x_2) + (2x_2) = 0$$

Since the equilibrium equations are satisfied, the body is in equilibrium.

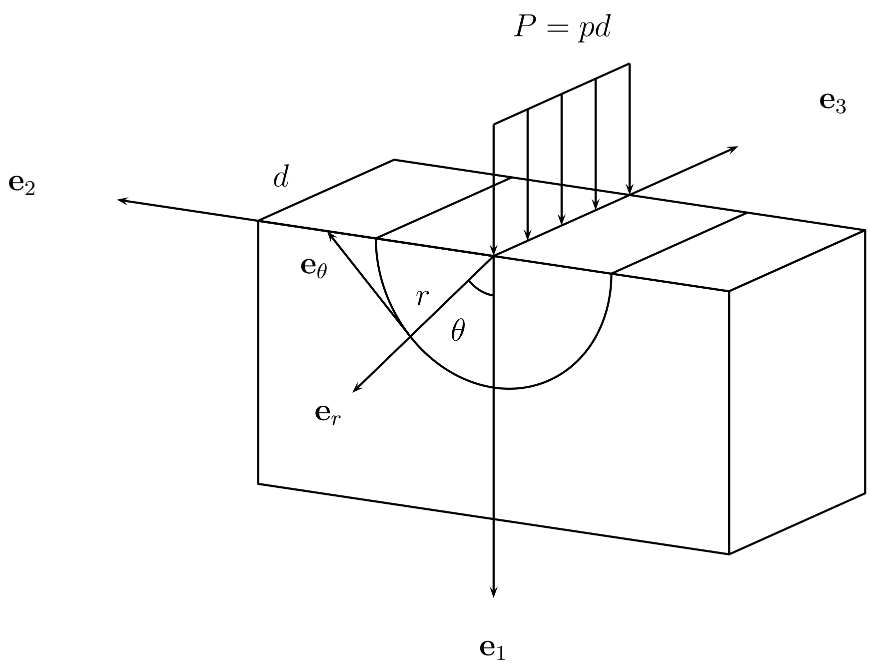
○ **Problems M-7.5** [7 points]
It is shown in higher-level classes on elasticity theory that the stress field in a semi-infinite plate of thickness d occupying the region ($x_1 \geq 0$; $-\infty \leq x_2 \leq +\infty$, $0 \leq x_3 \leq d$) with a concentrated normal edge load of magnitude $P = pd$, as shown below, has Cartesian scalar components $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$ and

$$\sigma_{11} = -\frac{2p\cos^3\theta}{\pi r} \tag{36}$$

$$\sigma_{22} = -\frac{2p\sin^2\theta\cos\theta}{\pi r} \tag{37}$$

$$\sigma_{12} = -\frac{2p\sin\theta\cos^2\theta}{\pi r} \tag{38}$$

where r and θ are the cylindrical coordinates. Note that in this case, the cartesian components in the given basis $\mathbf{e}_i$ are provided and that they are expressed as a function of the cylindrical coordinates r, θ.



5.1 (2 points) Determine the components of stress in the cylindrical coordinate system, i.e. $\sigma_{rr}(r,\theta)$, $\sigma_{\theta\theta}(r,\theta)$, $\sigma_{r\theta}(r,\theta)$ (Hint: use the 2D stress transformation equations). Interpret your result with the aid of a sketch of a cylindrical surface centered at the origin, and drawing a few material elements on the surface aligned with the radial and hoop directions together with the stress components acting on the planes with those normals. What can you say in terms of: principal stresses and directions, and shear stresses acting on any cylindrical surface centered at the point of application of the load?

Solution: Using the transformation equation,

$$\sigma_{rr} = \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\cos\theta\sin\theta = -\frac{2p\cos\theta}{\pi r} \tag{39}$$

$$\sigma_{\theta\theta} = \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\sigma_{12}\cos\theta\sin\theta = 0 \tag{40}$$

$$\sigma_{r\theta} = (\sigma_{22} - \sigma_{11})\sin\theta\cos\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta) = 0 \tag{41}$$

The state of stresses reduces to one single stress component in the radial direction: σ_{rr}. The principal direction is $\mathbf{e}_r$ and the principal stress is σ_{rr}. The shear stresses acting on any cylindrical surface centered at the point of application of the load are 0.

5.2 (2 points) Show that the body is in equilibrium (Hint: use the differential equations of stress equilibrium in cylindrical coordinates derived in problem 3).

Solution: Using the results from M7.1(f), the equilibrium equation in cylindrical system reads,

$$\begin{aligned}\nabla\cdot\boldsymbol{\sigma} + \mathbf{f} = &\left(\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}\right)\mathbf{e}_r \\ &+ \left(\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r}\right)\mathbf{e}_\theta \\ &+ \left(\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r}\right)\mathbf{e}_z \\ &+ f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z \\ = &\ \mathbf{0}\end{aligned} \tag{42}$$

Substitute $\sigma_{rr} = \frac{-2p\cos\theta}{\pi r}$, $\sigma_{\theta\theta} = 0$ and $\sigma_{r\theta} = 0$, we have

$$\nabla\cdot\boldsymbol{\sigma} + \mathbf{f} = \left(\frac{\partial(\frac{-2p\cos\theta}{\pi r})}{\partial r} + \frac{\frac{-2p\cos\theta}{\pi r}}{r}\right)\mathbf{e}_r$$

$$= (\frac{2p\cos\theta}{\pi r^2} - \frac{2p\cos\theta}{\pi r^2})\mathbf{e}_r \tag{43}$$

$$= \mathbf{0} \tag{44}$$

5.3 (1 point) Derive an expression for the traction vector acting at points on the semi-cylindrical surface $r = a$.

Solution: The traction $\mathbf{t}$ is

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = (\sigma_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{r\theta}\mathbf{e}_r \otimes \mathbf{e}_\theta + \sigma_{\theta r}\mathbf{e}_\theta \otimes \mathbf{e}_r) \cdot \mathbf{e}_r = \sigma_{rr}\mathbf{e}_r + \sigma_{\theta r}\mathbf{e}_\theta = -\frac{2p\cos\theta}{\pi r}\mathbf{e}_r \tag{45}$$

5.4 (2 points) Show that the traction distribution on the semicylindrical surface $r = a$ is in equilibrium with the edge load, for arbitrary a. To do this, you will need to calculate the resultant forces and moments due to the traction distribution by integrating over the surface.

Solution: The resultant forces (in x_1 and x_2-directions) due to the traction at $r = a$ are

$$F_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{t} \cdot \mathbf{e}_1 a d\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{2p\cos\theta}{\pi r}\mathbf{e}_r \cdot \mathbf{e}_1 a d\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{2p\cos\theta}{\pi r}\cos\theta a d\mathrm{d}\theta = -pd \quad (46)$$

and

$$F_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{t} \cdot \mathbf{e}_2 a d\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{2p\cos\theta}{\pi r}\sin\theta a d\mathrm{d}\theta = 0 \quad (47)$$

Thus the traction is in equilibrium with the applied load $P = pd$.