16.001 - Materials & Structures
Problem Set #5

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○ **Problem M-5.1**

(M.O. M8)

Let's begin with some practice on applying indicial notation. Evaluate the following expressions (where δ_{ij} is the Kronecker delta and ϵ_{ijk} is the permutation tensor):

(a) $\delta_{ij}\delta_{ij}$

Solution:

$$\delta_{ij}\delta_{ij} = \delta_{ij}\delta_{ji} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = \boxed{3}$$

where we have used the "index replacing" property of the kronecker delta.

(b) $\epsilon_{ijk}\epsilon_{kij}$ (*Hint: Expand this expression with all possible values for i, j, and k*)

Solution: Considering all possibilities of $\epsilon_{ijk}\epsilon_{kij}$ with distinct i, j, k

$$\begin{aligned} \epsilon_{ijk}\epsilon_{kij} &= \epsilon_{123}\epsilon_{312} + \epsilon_{132}\epsilon_{213} + \epsilon_{213}\epsilon_{321} + \epsilon_{231}\epsilon_{123} + \epsilon_{312}\epsilon_{231} + \epsilon_{321}\epsilon_{132} \\ &= (1)(1) + (-1)(-1) + (-1)(-1) + (1)(1) + (1)(1) + (-1)(-1) \\ &= 1+1+1+1+1+1 \\ &= \boxed{6} \end{aligned}$$

(c) $\delta_{ij}\epsilon_{ijk}$

Solution: By the index replacing property of kronecker delta, replace either i or j in ϵ_{ijk} with the other index so that

$$\delta_{ij}\epsilon_{ijk} = \epsilon_{iik} \text{ or } \epsilon_{jjk} = \boxed{0} \tag{1}$$

since ϵ_{ijk} is nonzero only if i, j and k were distinct.

(d) $\delta_{ij}\delta_{ik}\delta_{jk}$

Solution: Rearranging the δs and indices:

$$\delta_{ij}\delta_{ik}\delta_{jk} = \delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ii} = \boxed{3} \tag{2}$$

(e) Finally, show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$
(*Hint: Represent these expressions in indicial notation, using what was derived for the cross and dot products in class*)

Solution: In indicial notation:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_j b_k \cdot \mathbf{c} = \epsilon_{ijk} a_j b_k c_i \quad (3)$$

$$(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \epsilon_{ijk} b_j c_k \cdot \mathbf{a} = \epsilon_{ijk} a_i b_j c_k \quad (4)$$

$$(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = \epsilon_{ijk} c_j a_k \cdot \mathbf{b} = \epsilon_{ijk} a_k b_i c_j \quad (5)$$

$$(6)$$

We need to show that all of the expressions in indicial notation are equal to each other. We will show that the second and third expressions are equal to the first, and thereby all are equivalent.

- $\epsilon_{ijk} a_i b_j c_k$ (2nd expression) Making the swaps $i \to j$, $j \to k$, $k \to i$

  $$\epsilon_{ijk} a_i b_j c_k = \epsilon_{jki} a_j b_k c_i \quad (7)$$

  However, $\epsilon_{jki} = \epsilon_{ijk}$ since both expressions have the same permutation of i, j, k. Thus,

  $$\epsilon_{jki} a_j b_k c_i = \epsilon_{ijk} a_j b_k c_i \quad (8)$$

  which is exactly the same as the first expression.

- $\epsilon_{ijk} a_k b_i c_j$ (3rd expression) Now, making the swaps $i \to k$, $j \to i$, $k \to j$

  $$\epsilon_{ijk} a_k b_i c_j = \epsilon_{kij} a_j b_k c_i \quad (9)$$

  However, $\epsilon_{kij} = \epsilon_{ijk}$ since both expressions have the same permutation of i, j, k. Thus,

  $$\epsilon_{kij} a_j b_k c_i = \epsilon_{ijk} a_j b_k c_i \quad (10)$$

  which is exactly the same as the first expression. Thus, all three expressions are equivalent, so we have shown that

  $$\boxed{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}} \quad (11)$$

Problem M-5.2

In relation to the Cauchy tetrahedron discussed in class (shown in Figure 1 and found in the lecture notes), express in words in no more than a sentence or two what concepts the following quantities represent:

(a) $\mathbf{t}^{(\mathbf{n})}$

(b) $t_n = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$

(c) $\mathbf{t}_s = \mathbf{t}^{(\mathbf{n})} - t_n\mathbf{n}$

(d) $t_s = |\mathbf{t}_s|$

Solution:

(a) The quantity $\mathbf{t}^{(\mathbf{n})}$ represents the stress vector acting on a plane with a unit normal vector given by $\mathbf{n}$.

(b) The quantity t_n is a scalar value indicating the component of the stress vector acting along the normal direction $\mathbf{n}$. This represents the normal stress on the plane.

(c) The quantity $\mathbf{t}_s$ represents the projection of the stress vector $\mathbf{t}^{(\mathbf{n})}$ onto the plane with unit normal $\mathbf{n}$.

(d) The quantity t_s is a scalar value indicating the magnitude of the stress vector acting in the direction of the plane with normal $\mathbf{n}$. This represents the shear stress on the plane.

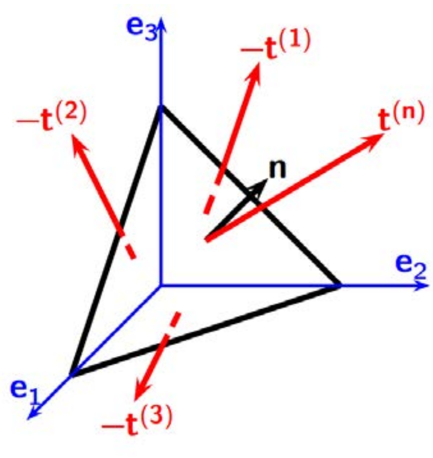

Figure 1: Cauchy tetrahedron.

○ **Problem M-5.3**

(M.O. M8)

In the Cauchy tetrahedron, consider a plane with unit normal $\mathbf{n} = 1/\sqrt{3}\mathbf{e_1} + 1/\sqrt{3}\mathbf{e_2} + 1/\sqrt{3}\mathbf{e_3}$. In this plane, commonly known as an octahedral plane, the stress tensor $\boldsymbol{\sigma}$ has the following representation

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

where σ_1, σ_2 and σ_3 are the principal stresses and $\mathbf{n}$ is the principal direction (we will cover principal stresses and directions and how to find them later on in the class). For this octahedral plane, determine:

(a) The magnitude of the normal stress on the plane.

(b) The magnitude of the resultant shear stress on the plane.

Solution:

(a) We must obtain the stress vector $\mathbf{t}^{(\mathbf{n})}$ on this plane and then the normal stress $t_n = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$. From the notes on Stress (slides 22-23) we have that $\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma}$ or ${t_j}^{(\mathbf{n})} = \sigma_{ij} n_i$. Therefore, the components of the stress vector on this plane are:

$$\begin{aligned} {t_1}^{(\mathbf{n})} &= (\sigma_{11} + \sigma_{21} + \sigma_{31}) n_1 = \frac{1}{\sqrt{3}} \sigma_1 \\ {t_2}^{(\mathbf{n})} &= (\sigma_{12} + \sigma_{22} + \sigma_{32}) n_2 = \frac{1}{\sqrt{3}} \sigma_2 \\ {t_3}^{(\mathbf{n})} &= (\sigma_{13} + \sigma_{23} + \sigma_{33}) n_3 = \frac{1}{\sqrt{3}} \sigma_3 \end{aligned}$$

Overall $\mathbf{t}^{(\mathbf{n})} = \frac{1}{\sqrt{3}} (\sigma_1 \mathbf{e_1} + \sigma_2 \mathbf{e_2} + \sigma_3 \mathbf{e_3})$. Now we can obtain $t_n = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$:

$$t_n = \frac{1}{\sqrt{3}} (\sigma_1 \mathbf{e_1} + \sigma_2 \mathbf{e_2} + \sigma_3 \mathbf{e_3}) \cdot \frac{1}{\sqrt{3}} (\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) = \boxed{\frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)}$$

(b) The resultant shear stress is given by $t_s = |\mathbf{t}^{(\mathbf{n})} - t_n \mathbf{n}|$.

$$\begin{aligned} &\mathbf{t}^{(\mathbf{n})} - t_n \mathbf{n} = \\ &\frac{1}{\sqrt{3}} (\sigma_1 \mathbf{e_1} + \sigma_2 \mathbf{e_2} + \sigma_3 \mathbf{e_3}) - \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) \left(\frac{1}{\sqrt{3}} (\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) \right) = \\ &\frac{1}{3\sqrt{3}} (2\sigma_1 - \sigma_2 - \sigma_3) \mathbf{e_1} + \frac{1}{3\sqrt{3}} (2\sigma_2 - \sigma_1 - \sigma_3) \mathbf{e_2} + \frac{1}{3\sqrt{3}} (2\sigma_3 - \sigma_1 - \sigma_2) \mathbf{e_3} \end{aligned}$$

upon performing some algebra. Taking the magnitude of the resultant shear stress vector found above

$$t_s = \sqrt{\frac{((2\sigma_1 - \sigma_2 - \sigma_3))^2 + ((2\sigma_2 - \sigma_1 - \sigma_3))^2 + ((2\sigma_3 - \sigma_1 - \sigma_2))^2}{27}}$$

○ **Problem M-5.4**

For the set of stress vectors $\mathbf{t}^{(i)}$ given in terms of their components in the cartesian basis $\mathbf{e}_i$:

$$\mathbf{t}^{(1)} = 1\mathbf{e}_1 + 0\mathbf{e}_2 \tag{12}$$

$$\mathbf{t}^{(2)} = 0\mathbf{e}_1 - 1\mathbf{e}_2 \tag{13}$$

$$\mathbf{t}^{(3)} = \mathbf{0} \tag{14}$$

(a) (2 pts) Compute each one of the quantities in Problem M-5.2 for a normal unit vector $\mathbf{n}$ forming an angle α with the $\mathbf{e}_1$ axis: $\mathbf{n} = \cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2 + 0\mathbf{e}_3$

Solution:

- $\mathbf{t}^{(\mathbf{n})}$

$$\begin{aligned}\mathbf{t}^{(\mathbf{n})} &= \mathrm{n}_i\mathbf{t}^{(i)} \\ &= \mathrm{n}_1\mathbf{t}^{(1)} + \mathrm{n}_2\mathbf{t}^{(2)} + \mathrm{n}_3\mathbf{t}^{(3)} \\ &= \cos(\alpha)(1\mathbf{e}_1 + 0\mathbf{e}_2) + \sin(\alpha)(0\mathbf{e}_1 - 1\mathbf{e}_2) + 0(0\mathbf{e}_3) \\ &= 1\cos(\alpha)\mathbf{e}_1 - 1\sin(\alpha)\mathbf{e}_2 + 0\mathbf{e}_3 \\ &= \boxed{\cos(\alpha)\mathbf{e}_1 - \sin(\alpha)\mathbf{e}_2}\end{aligned}$$

- $t_n = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$

$$\begin{aligned}t_n &= \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} \\ &= (\cos(\alpha)\mathbf{e}_1 - \sin(\alpha)\mathbf{e}_2 + 0\mathbf{e}_3) \cdot (\cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2 + 0\mathbf{e}_3) \\ &= (\cos^2(\alpha) - \sin^2(\alpha)) \\ &= \boxed{\cos(2\alpha)}\end{aligned}$$

- $\mathbf{t}_s = \mathbf{t}^{(\mathbf{n})} - t_n\mathbf{n}$

$$\begin{aligned}\mathbf{t}_s &= \mathbf{t}^{(\mathbf{n})} - t_n\mathbf{n} \\ &= (\cos(\alpha)\mathbf{e}_1 - \sin(\alpha)\mathbf{e}_2) - \cos(2\alpha)(\cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2) \\ &= \cos(\alpha)(1 - \cos(2\alpha))\mathbf{e}_1 - \sin(\alpha)(1 + \cos(2\alpha))\mathbf{e}_2 \\ &= \cos(\alpha)(2\sin^2(\alpha))\mathbf{e}_1 - \sin(\alpha)(2\cos^2(\alpha))\mathbf{e}_2 \\ &= 2\cos(\alpha)\sin(\alpha)(\sin(\alpha)\mathbf{e}_1 - \cos(\alpha)\mathbf{e}_2) \\ &= \boxed{\sin(2\alpha)(\sin(\alpha)\mathbf{e}_1 - \cos(\alpha)\mathbf{e}_2)}\end{aligned}$$

- $t_s = |\mathbf{t}_s|$

$$\begin{aligned}t_s &= |\mathbf{t}_s| \\ &= \sin(2\alpha)|(\sin(\alpha)\mathbf{e}_1 - \cos(\alpha)\mathbf{e}_2)| \\ &= \sin(2\alpha)\sqrt{\sin^2(\alpha) + (-\cos^2(\alpha))} \\ &= \boxed{\sin(2\alpha)}\end{aligned}$$

(b) (1 pt) Find $\mathbf{n}^*$ such that $t_s^* = 0$. Then compute t_n^*.

Solution: Using the expression for t_s found in part (a)

$$t_s^* = 0$$
$$\sin(2\alpha) = 0$$

This is true for $2\alpha = 0, \pi, 2\pi ...$ or when $\boxed{\alpha = \frac{m\pi}{2}}$, where m is any integer. The corresponding normal is:

$$\boxed{\mathbf{n}^* = \cos\left(\frac{m\pi}{2}\right)\mathbf{e}_1 + \sin\left(\frac{m\pi}{2}\right)\mathbf{e}_2}$$

For this normal, we find that t_n^* is given by

$$t_n^* = \cos(2\alpha) = \boxed{\cos(m\pi)} = \boxed{\begin{cases} 1 & \text{if } m \text{ is even,} \\ -1 & \text{if } m \text{ is odd.} \end{cases}}$$

(c) (1 pt) Find $\mathbf{n}^{**}$ such that t_n^{**} is the maximum of t_n for all $\mathbf{n}$. Then compute t_n^{**} and t_s^{**}.

Solution: Using the expression for t_n found in part (a), we know that $t_n = \cos(2\alpha)$ is maximized when $\cos(2\alpha) = 1$, which occurs when $\alpha = 0, \pi, 2\pi$, or more generally when $\boxed{\alpha = m\pi}$ for any integer m. The corresponding normal is:

$$\boxed{\mathbf{n}^{**} = \cos(m\pi)\mathbf{e}_1 + \sin(m\pi)\mathbf{e}_2} = \boxed{\begin{cases} \mathbf{e}_1 & \text{if } m \text{ is even,} \\ -\mathbf{e}_1 & \text{if } m \text{ is odd,}. \end{cases}}$$

The corresponding t_n^{**} and t_s^{**} are then

$$\boxed{t_n^{**} = 1}$$
$$\boxed{t_s^{**} = \sin(2\alpha) = 1\sin(2m\pi) = 0}$$

(d) (1 pt) Find $\mathbf{n}^{***}$ such that t_s^{***} is the maximum of t_s for all $\mathbf{n}$. Then compute t_s^{***} and t_n^{***}.

Solution: Using the expression found for t_s found in part (a), we know that $t_s = \sin(2\alpha)$ is maximized when $\sin(2\alpha) = 1$, which occurs when $\alpha = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, ...,$ or more generally when $\boxed{\alpha = \frac{\pi}{4} + m\pi}$ for any integer m. The corresponding normal is:

$$\boxed{\mathbf{n}^{***} = \cos\left(\frac{\pi}{4} + m\pi\right)\mathbf{e}_1 + \sin\left(\frac{\pi}{4} + m\pi\right)\mathbf{e}_2} = \boxed{\begin{cases} \frac{\sqrt{2}}{2}(\mathbf{e}_1 + \mathbf{e}_2) & \text{if } m \text{ is even,} \\ \frac{\sqrt{-2}}{2}(\mathbf{e}_1 + \mathbf{e}_2) & \text{if } m \text{ is odd, .} \end{cases}}$$

The corresponding t_s^{***} and t_s^{**} are then

$$\boxed{t_s^{***} = 1}$$

$$\boxed{t_n^{***} = 1\cos(2\alpha) = 1\cos\left(\frac{\pi}{2} + 2m\pi\right) = 0}$$

○ **Problem M-5.5**

The state of stress at a point is shown on a material element in terms of the stress vectors $\mathbf{t}^{(i)}$, see Figure 2. The numerical values are given in MPa.

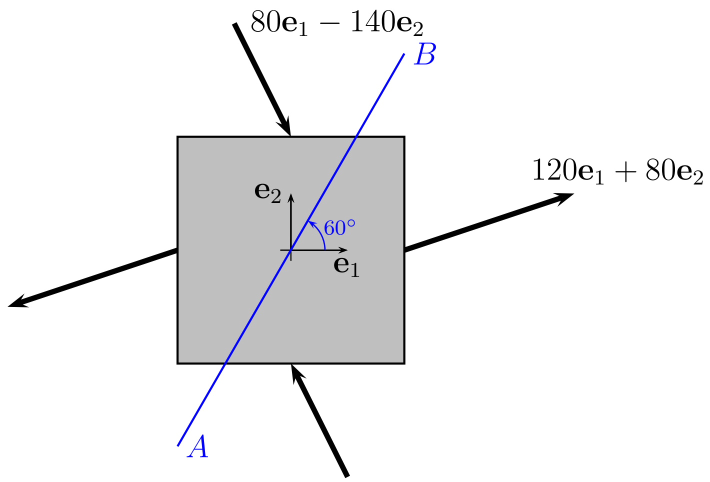

Figure 2: State of stress at a point on a material element.

On the inclined plane AB, determine:

(a) (1pt) The total stress $\mathbf{t}^{(\mathbf{n})}$

Solution: Using vector equations derived in class:

$$\mathbf{t}^{(\mathbf{n})} = n_i \mathbf{t}^{(i)}$$

$$\mathbf{n} = \cos(-30^\circ)\mathbf{e}_1 + \sin(-30^\circ)\mathbf{e}_2 = \frac{\sqrt{3}}{2}\mathbf{e}_1 - \frac{1}{2}\mathbf{e}_2$$

$$\mathbf{t}^{(\mathbf{n})} = \frac{\sqrt{3}}{2}\underbrace{(120\mathbf{e}_1 + 80\mathbf{e}_2)}_{\mathbf{t}^{(1)}} - \frac{1}{2}\underbrace{(80\mathbf{e}_1 - 140\mathbf{e}_2)}_{\mathbf{t}^{(2)}}$$

$$= \left(\sqrt{3}\times 60 - 40\right)\mathbf{e}_1 + \left(\sqrt{3}\times 40 + 70\right)\mathbf{e}_2 = 10\left[\left(6\sqrt{3} - 4\right)\mathbf{e}_1 + \left(4\sqrt{3} + 7\right)\mathbf{e}_2\right]$$

(b) (1pt) The normal stress t_n

Solution:

$$t^n = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = 10\left[\left(6\sqrt{3} - 4\right)\frac{\sqrt{3}}{2} + \left(4\sqrt{3} + 7\right)\frac{-1}{2}\right]$$

$$\rightarrow \boxed{t^n = 10\left(\frac{11}{2} - 4\sqrt{3}\right)\text{MPa} \sim -14\text{MPa}}$$

(c) (1pt) The shear stress t_s

Solution:

$$t^s = \|\mathbf{t}^{(\mathbf{n})} - t^n \mathbf{n}\| = \|10\left(2 + \frac{13\sqrt{3}}{4}\right)\mathbf{e}_1 + 10\left(\frac{39}{4} + 2\sqrt{3}\right)\mathbf{e}_2\|$$

$$\rightarrow \boxed{t^s \sim 152\text{MPa}}$$