# 16.001 - Materials & Structures
# Problem Set #5

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**Problem M-5.1**
(M.O. M8)
Let's begin with some practice on applying indicial notation. Evaluate the following expressions (where $\delta_{ij}$ is the Kronecker delta and $\epsilon_{ijk}$ is the permutation tensor):

(a) $\delta_{ij}\delta_{ij}$

**Solution:**

$$\delta_{ij}\delta_{ij} = \delta_{ij}\delta_{ji} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = \boxed{3}$$

where we have used the "index replacing" property of the kronecker delta.

(b) $\epsilon_{ijk}\epsilon_{kij}$ (*Hint: Expand this expression with all possible values for i, j, and k*)

**Solution:** Considering all possibilities of $\epsilon_{ijk}\epsilon_{kij}$ with distinct $i, j, k$

$$\begin{aligned}\epsilon_{ijk}\epsilon_{kij} &= \epsilon_{123}\epsilon_{312} + \epsilon_{132}\epsilon_{213} + \epsilon_{213}\epsilon_{321} + \epsilon_{231}\epsilon_{123} + \epsilon_{312}\epsilon_{231} + \epsilon_{321}\epsilon_{132} \\ &= (1)(1) + (-1)(-1) + (-1)(-1) + (1)(1) + (1)(1) + (-1)(-1) \\ &= 1+1+1+1+1+1 \\ &= \boxed{6}\end{aligned}$$

(c) $\delta_{ij}\epsilon_{ijk}$

**Solution:** By the index replacing property of kronecker delta, replace either $i$ or $j$ in $\epsilon_{ijk}$ with the other index so that

$$\delta_{ij}\epsilon_{ijk} = \epsilon_{iik} \text{ or } \epsilon_{jjk} = \boxed{0} \tag{1}$$

since $\epsilon_{ijk}$ is nonzero only if $i, j$ and $k$ were distinct.

(d) $\delta_{ij}\delta_{ik}\delta_{jk}$

**Solution:** Rearranging the $\delta$s and indices:

$$\delta_{ij}\delta_{ik}\delta_{jk} = \delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ii} = \boxed{3} \tag{2}$$

(e) Finally, show that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$
(*Hint: Represent these expressions in indicial notation, using what was derived for the cross and dot products in class*)

**Solution:** In indicial notation:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_j b_k \cdot \mathbf{c} = \epsilon_{ijk} a_j b_k c_i \tag{3}$$
$$(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \epsilon_{ijk} b_j c_k \cdot \mathbf{a} = \epsilon_{ijk} a_i b_j c_k \tag{4}$$
$$(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = \epsilon_{ijk} c_j a_k \cdot \mathbf{b} = \epsilon_{ijk} a_k b_i c_j \tag{5}$$
$$\tag{6}$$

We need to show that all of the expressions in indicial notation are equal to each other. We will show that the second and third expressions are equal to the first, and thereby all are equivalent.

- $\epsilon_{ijk} a_i b_j c_k$ (2nd expression) Making the swaps $i \to j$, $j \to k$, $k \to i$

  $$\epsilon_{ijk} a_i b_j c_k = \epsilon_{jki} a_j b_k c_i \tag{7}$$

  However, $\epsilon_{jki} = \epsilon_{ijk}$ since both expressions have the same permutation of $i, j, k$. Thus,

  $$\epsilon_{jki} a_j b_k c_i = \epsilon_{ijk} a_j b_k c_i \tag{8}$$

  which is exactly the same as the first expression.

- $\epsilon_{ijk} a_k b_i c_j$ (3rd expression) Now, making the swaps $i \to k$, $j \to i$, $k \to j$

  $$\epsilon_{ijk} a_k b_i c_j = \epsilon_{kij} a_j b_k c_i \tag{9}$$

  However, $\epsilon_{kij} = \epsilon_{ijk}$ since both expressions have the same permutation of $i, j, k$. Thus,

  $$\epsilon_{kij} a_j b_k c_i = \epsilon_{ijk} a_j b_k c_i \tag{10}$$

  which is exactly the same as the first expression. Thus, all three expressions are equivalent, so we have shown that

  $$\boxed{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}} \tag{11}$$

**Problem M-5.2**

In relation to the Cauchy tetrahedron discussed in class (shown in Figure 1 and found in the lecture notes), express in words in no more than a sentence or two what concepts the following quantities represent:

(a) $\mathbf{t}^{(\mathbf{n})}$

(b) $t_n = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$

(c) $\mathbf{t}_s = \mathbf{t}^{(\mathbf{n})} - t_n\mathbf{n}$

(d) $t_s = |\mathbf{t}_s|$

**Solution:**

(a) The quantity $\mathbf{t}^{(\mathbf{n})}$ represents the stress vector acting on a plane with a unit normal vector given by $\mathbf{n}$.

(b) The quantity $t_n$ is a scalar value indicating the component of the stress vector acting along the normal direction $\mathbf{n}$. This represents the normal stress on the plane.

(c) The quantity $\mathbf{t}_s$ represents the projection of the stress vector $\mathbf{t}^{(\mathbf{n})}$ onto the plane with unit normal $\mathbf{n}$.

(d) The quantity $t_s$ is a scalar value indicating the magnitude of the stress vector acting in the direction of the plane with normal $\mathbf{n}$. This represents the shear stress on the plane.

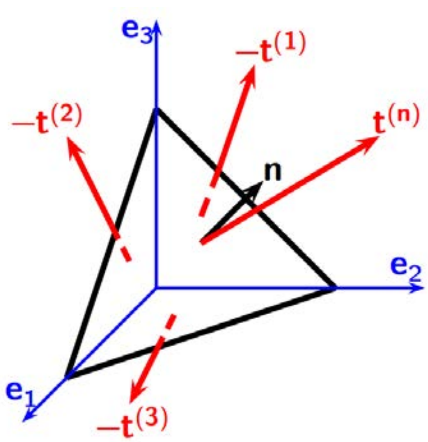


Figure 1: Cauchy tetrahedron.

**Problem M-5.3**
(M.O. M8)
In the Cauchy tetrahedron, consider a plane with unit normal $\mathbf{n} = 1/\sqrt{3}\mathbf{e_1} + 1/\sqrt{3}\mathbf{e_2} + 1/\sqrt{3}\mathbf{e_3}$. In this plane, commonly known as an octahedral plane, the stress tensor $\boldsymbol{\sigma}$ has the following representation

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the principal stresses and $\mathbf{n}$ is the principal direction (we will cover principal stresses and directions and how to find them later on in the class). For this octahedral plane, determine:

(a) The magnitude of the normal stress on the plane.

(b) The magnitude of the resultant shear stress on the plane.

**Solution:**

(a) We must obtain the stress vector $\mathbf{t}^{(\mathbf{n})}$ on this plane and then the normal stress $t_n = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$. From the notes on Stress (slides 22-23) we have that $\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma}$ or ${t_j}^{(\mathbf{n})} = \sigma_{ij} n_i$. Therefore, the components of the stress vector on this plane are:

$$\begin{aligned} {t_1}^{(\mathbf{n})} &= (\sigma_{11} + \sigma_{21} + \sigma_{31}) n_1 = \frac{1}{\sqrt{3}} \sigma_1 \\ {t_2}^{(\mathbf{n})} &= (\sigma_{12} + \sigma_{22} + \sigma_{32}) n_2 = \frac{1}{\sqrt{3}} \sigma_2 \\ {t_3}^{(\mathbf{n})} &= (\sigma_{13} + \sigma_{23} + \sigma_{33}) n_3 = \frac{1}{\sqrt{3}} \sigma_3 \end{aligned}$$

Overall $\mathbf{t}^{(\mathbf{n})} = \frac{1}{\sqrt{3}} (\sigma_1 \mathbf{e_1} + \sigma_2 \mathbf{e_2} + \sigma_3 \mathbf{e_3})$. Now we can obtain $t_n = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$:

$$t_n = \frac{1}{\sqrt{3}} (\sigma_1 \mathbf{e_1} + \sigma_2 \mathbf{e_2} + \sigma_3 \mathbf{e_3}) \cdot \frac{1}{\sqrt{3}} (\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) = \boxed{\frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)}$$

(b) The resultant shear stress is given by $t_s = |\mathbf{t}^{(\mathbf{n})} - t_n \mathbf{n}|$.

$$\begin{aligned} \mathbf{t}^{(\mathbf{n})} - t_n \mathbf{n} &= \\ \frac{1}{\sqrt{3}} (\sigma_1 \mathbf{e_1} + \sigma_2 \mathbf{e_2} + \sigma_3 \mathbf{e_3}) - \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) \left( \frac{1}{\sqrt{3}} (\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}) \right) &= \\ \frac{1}{3\sqrt{3}} (2\sigma_1 - \sigma_2 - \sigma_3) \mathbf{e_1} + \frac{1}{3\sqrt{3}} (2\sigma_2 - \sigma_1 - \sigma_3) \mathbf{e_2} + \frac{1}{3\sqrt{3}} (2\sigma_3 - \sigma_1 - \sigma_2) \mathbf{e_3} \end{aligned}$$

upon performing some algebra. Taking the magnitude of the resultant shear stress vector found above

$$\boxed{t_s = \sqrt{\frac{((2\sigma_1 - \sigma_2 - \sigma_3))^2 + ((2\sigma_2 - \sigma_1 - \sigma_3))^2 + ((2\sigma_3 - \sigma_1 - \sigma_2))^2}{27}}}$$

○ **Problem M-5.4**

For the set of stress vectors $\mathbf{t}^{(i)}$ given in terms of their components in the cartesian basis $\mathbf{e}_i$:

$$\mathbf{t}^{(1)} = 1\mathbf{e}_1 + 0\mathbf{e}_2 \tag{12}$$

$$\mathbf{t}^{(2)} = 0\mathbf{e}_1 - 1\mathbf{e}_2 \tag{13}$$

$$\mathbf{t}^{(3)} = \mathbf{0} \tag{14}$$

(a) (2 pts) Compute each one of the quantities in Problem M-5.2 for a normal unit vector $\mathbf{n}$ forming an angle $\alpha$ with the $\mathbf{e}_1$ axis: $\mathbf{n} = \cos(\alpha)\mathbf{e}_1 + \sin(\alpha)\mathbf{e}_2 + 0\mathbf{e}_3$

(b) (1 pt) Find $\mathbf{n}^*$ such that $t_s^* = 0$. Then compute $t_n^*$.

(c) (1 pt) Find $\mathbf{n}^{**}$ such that $t_n^{**}$ is the maximum of $t_n$ for all $\mathbf{n}$. Then compute $t_n^{**}$ and $t_s^{**}$.

(d) (1 pt) Find $\mathbf{n}^{***}$ such that $t_s^{***}$ is the maximum of $t_s$ for all $\mathbf{n}$. Then compute $t_s^{***}$ and $t_n^{***}$.

**Problem M-5.5**

The state of stress at a point is shown on a material element in terms of the stress vectors $\mathbf{t}^{(i)}$, see Figure 2. The numerical values are given in MPa.

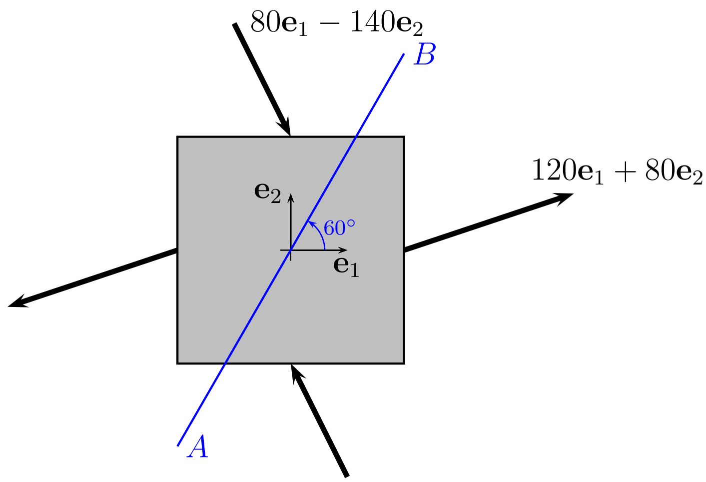


Figure 2: State of stress at a point on a material element.

On the inclined plane AB, determine:

(a) (1pt) The total stress $\mathbf{t}^{(\mathbf{n})}$

(b) (1pt) The normal stress $t_n$

(c) (1pt) The shear stress $t_s$

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