16.001 - Materials & Structures
Problem Set #4

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Problems M-4.1

(M.O M7,M8) Consider the rigid bar A-C shown in Figure 1. The bar is subject to a load of intensity P. The hinge support includes a torsional spring which reacts with a moment $M = k_T\theta$, where θ is the angle of rotation of the bar at that point and k_T is the torsional spring constant. The linear springs at the remaining supports have stiffnesses of k and $2k$, respectively.

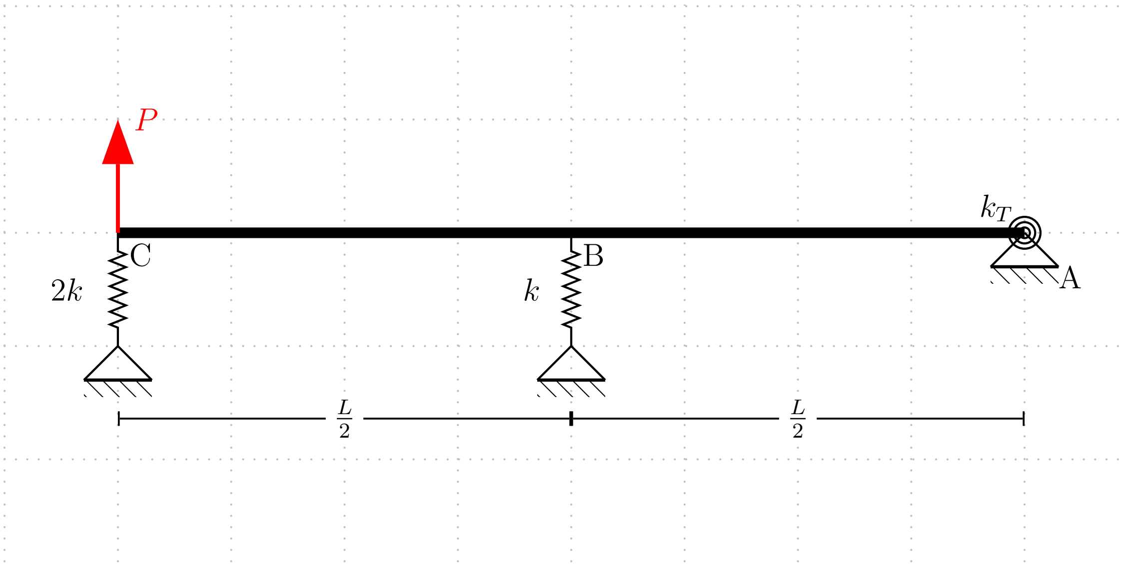


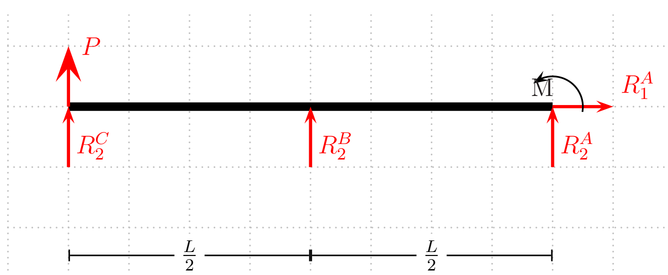
Figure 1

(a) (1 point) Is this structure statically determinate, indeterminate or unstable?

Solution: Indeterminate

(b) (2 points) Draw the FBD for the whole structure and write the equations of global equilibrium to obtain expressions for the unknown reactions.

Solution:



FBD shown in figure. From the FBD write the equations of equilibrium.

$$\sum F_1 = 0 = R_1^A$$

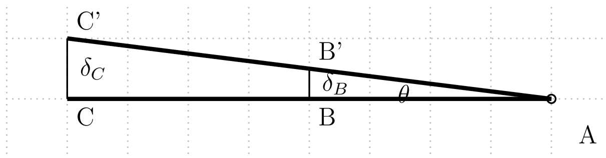
$$\sum F_2 = 0 = R_2^A + R_2^B + R_2^C + P$$

$$\sum M_A = 0 = -R_2^B \frac{L}{2} - R_2^C L + M - PL$$

Where M is a function of θ ($M(\theta)$) and the reactions at B and C are functions of their respective displacements ($R_2^B(\delta_B)$ & $R_2^C(\delta_C)$)

(c) (2 points) Draw a schematic of the deformed structure under load, identify appropriate kinematic variables describing the deformation and establish compatibility equations relating these kinematic variables.

Solution:



The schematic of deformation in shown in the figure. The kinematic variables in the problem are θ, δ_B, & δ_C, which are the rotation of the beam about A, the vertical displacement of the beam at B, and the vertical displacement of the beam at C, respectively. Using the small angle approximation ($sin(\theta) = \theta$), the compatibility equations for the structure are:

$$2\delta_B = \delta_C$$

$$\delta_B = \theta \frac{L}{2}$$

$$\delta_C = \theta L$$

(d) (1 point) Complete the system of equations by providing suitable constitutive laws for each of the participating structural components of this structural system. Keep in mind that the bar itself is assumed completely rigid and therefore does not have a constitutive law.

Solution:
Three constitutive equations are needed for this problem (one for each spring). Compatibility has been included in the final term of the last two equations.

$$M = k_t \theta$$

$$R_2^B = -k\delta_B = -k\theta \frac{L}{2}$$

$$R_2^C = -2k\delta_C = -2k\theta L$$

(e) (2 points) Solve the system of equations you have obtained and compute the reaction forces and the deflections of each spring.

Solution:

To solve the system of equations, start by substituting the constitutive laws with the compatibility included into the equations of equilibrium.

$$\sum F_2 = 0 = R_2^A - k\theta\frac{L}{2} - 2k\theta L + P$$
$$\sum M_A = 0 = k\theta\frac{L}{2}\frac{L}{2} + 2k\theta LL + k_t\theta - \frac{L^2 p_0}{2}$$

Solve the moment equation of equilibrium for θ, which results in:

$$\theta = \frac{4PL}{9kL^2 + 4k_t}\,[\text{rad}]$$

Next, substitute θ back into the equations for $\delta_B,\ \delta_C,\ R_2^B,\ M,\ \&\ R_2^C$.

$$\delta_B = \frac{2PL^2}{9kL^2 + 4k_t}\,[\text{m}]$$
$$\delta_C = \frac{4PL^2}{9kL^2 + 4k_t}\,[\text{m}]$$
$$M = -k_t\frac{4PL}{9kL^2 + 4k_t}\,[\text{N-m}]$$
$$R_2^B = -\frac{4kPL}{9kL^2 + 4k_t}\,[\text{N}]$$
$$R_2^C = -\frac{4kPL^2}{9kL^2 + 4k_t}\,[\text{N}]$$

Finally, substitute the reactions in the equation of equilibrium for F_2 and solve for R_2^A

$$R_2^A = -P + \frac{4kPL}{9kL^2 + 4k_t} + \frac{4kPL^2}{9kL^2 + 4k_t}\,[\text{N}]$$

You now have the 4 reaction forces and the deflections of the springs (2 translational & 1 rotational).

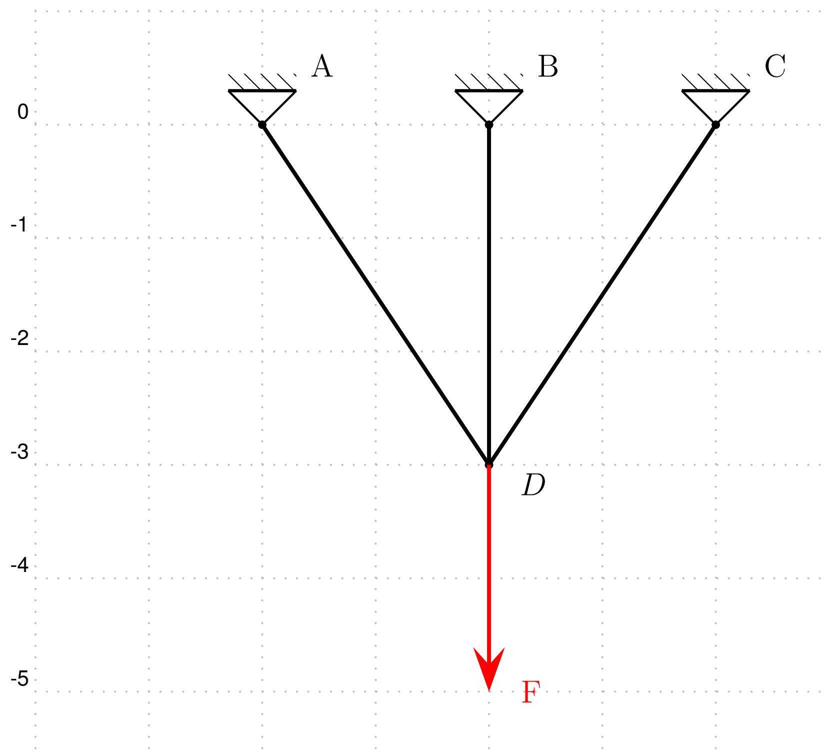


Figure 2

Problems M-4.2

(3 points) (MO: M7,M8) In the structure shown in Figure 2 the bars are made from aluminum tubes of Young's modulus E = 70GPa and cross-sectional area $A = 10mm^2$. The force F = 10KN. Additionally, the grid scale is in meters. Analyze the structure to find the following:

(a) The forces in each bar.

(b) The deflection of point D.

(c) The reactions at the supports.

Solution:

To start one must realize that this system is statically indeterminate. With that in mind, write the constitutive law and compatibility conditions.

Compatibility

$$\delta_{AD} = -\frac{3}{\sqrt{13}}u_2^D$$
$$\delta_{BD} = -u_2^D$$
$$\delta_{CD} = -\frac{3}{\sqrt{13}}u_2^D$$

Constitutive Law

$$F_{AD} = \frac{EA}{\sqrt{13}}\delta_{AD} = -\frac{3}{13}EAu_2^D$$
$$F_{BD} = \frac{EA}{3}\delta_{BD} = -\frac{EA}{3}u_2^D$$
$$F_{CD} = \frac{EA}{\sqrt{13}}\delta_{CD} = -\frac{3}{13}EAu_2^D$$

where positive u_2^D is defined in the positive e_2 direction.

Now,apply method of joints at point D. Specifically, looking at the vertical direction.

$$\begin{aligned}\sum F_2 = 0 &= -F + F_{BD} + \frac{3}{\sqrt{13}}F_CD + \frac{3}{\sqrt{13}}F_AD \\ &= -F + \left(-\frac{1}{3} - \frac{9}{13\sqrt{13}} - \frac{9}{13\sqrt{13}}\right)EAu_2^D \\ \rightarrow u_2^D &= -\frac{F}{\left(\frac{1}{3} + \frac{18}{13\sqrt{13}}\right)EA} = -0.020m\end{aligned}$$

Next, substitute the value for the deflection at D into the Constitutive law for each bar to find the forces in them.

$$F_{AD} = 3217N$$
$$F_{BD} = 4647N$$
$$F_{CD} = 3217N$$

Finally, solve for the reactions at the supports. By inspection of support B.

$$R_1^B = 0$$
$$R_2^B = 4647N$$

Applying method of joints at A.

$$\sum F_1 = 0 = R_1^A + \frac{2}{\sqrt{13}}F_{AD}$$
$$\sum F_2 = 0 = R_2^A - \frac{3}{\sqrt{13}}F_{AD}$$

solving for the reactions yield:

$$R_1^A = -\frac{2}{\sqrt{13}}F_{AD} = -1784.5N$$
$$R_2^A = \frac{3}{\sqrt{13}}F_{AD} = 2676.7N$$

Applying method of joints at C.

$$\sum F_1 = 0 = R_1^C - \frac{2}{\sqrt{13}}F_{CD}$$
$$\sum F_2 = 0 = R_2^C - \frac{3}{\sqrt{13}}F_{CD}$$

Solving for the reactions yield:

$$R_1^C = \frac{2}{\sqrt{13}}F_{CD} = 1784.5N$$
$$R_2^C = \frac{3}{\sqrt{13}}F_{CD} = 2676.7N$$

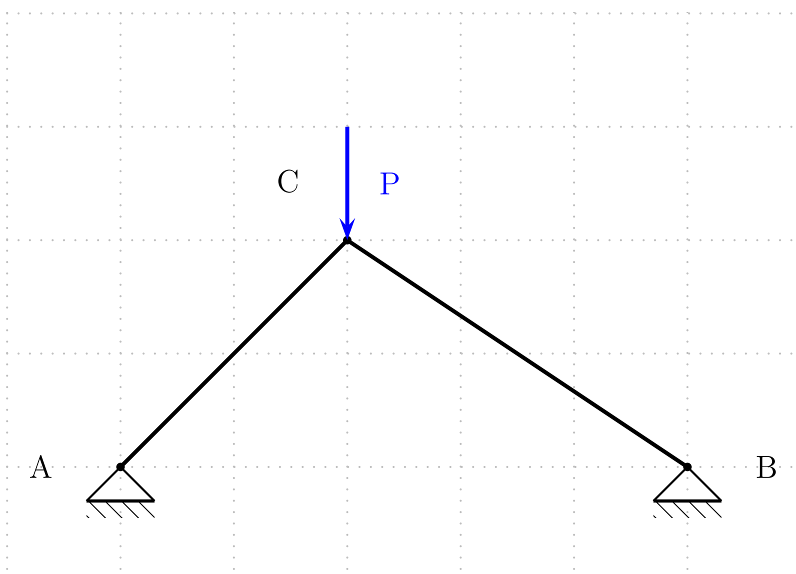


Figure 3: Free Body Diagram

Problems M-4.3

(4 points) Consider the linkage shown in Figure 4 comprised of two steel members ($E = 200$ GPa). An applied force of $P = 10$ kN is applied to the linkage at point C.

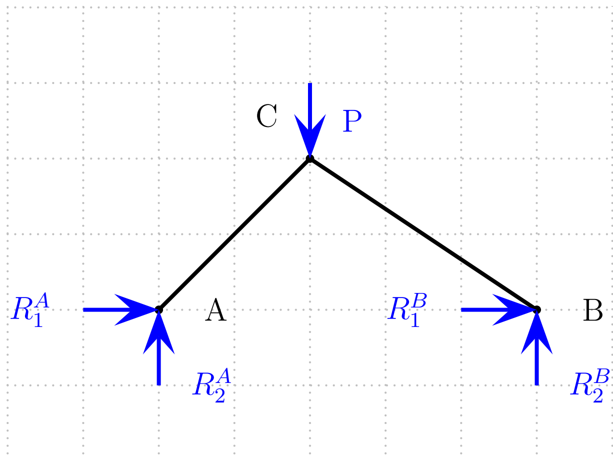
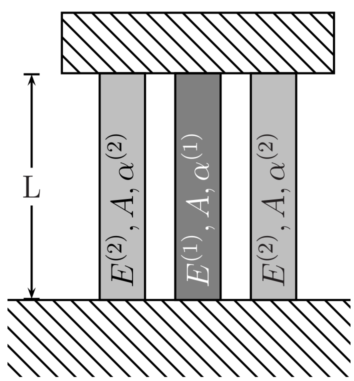


Figure 4: Two Member Steel Linkage

Determine the minimum allowable radius of the steel members such that point C does not have a vertical deflection of more than 1 cm. You may assume that the steel members have circular cross-sections, each of the same area. The grid scale is given in meters.

Solution: We begin by analyzing a global FBD of the linkage system



Applying force and moment equilibrium

$$\sum F_1 = 0 \rightarrow \; R_A^1 + R_B^1 = 0$$
$$\sum F_2 = 0 \rightarrow \; R_A^2 + R_B^2 - P = 0$$
$$\sum M^A = 0 \rightarrow \; -2P + 5R_B^2 = 0$$

We find that the vertical reactions at A and B are

$$R_2^B = \frac{2}{5}P$$
$$R_2^A = \frac{3}{5}P$$

Now applying method of joints at C

$$\sum F_1 = 0 \rightarrow \; -\frac{\sqrt{2}}{2}F_{AC} + \frac{3}{\sqrt{13}}F_{BC} = 0$$
$$\sum F_2 = 0 \rightarrow \; -\frac{\sqrt{2}}{2}F_{AC} - \frac{2}{\sqrt{13}}F_{BC} - P = 0$$

Solving yields

$$F_{AC} = \frac{-6}{5\sqrt{2}}P$$
$$F_{BC} = \frac{-\sqrt{13}}{5}P$$

Analyzing horizontal reactions at point A

$$\sum F_1^A = 0$$
$$R_1^A + \frac{\sqrt{2}}{2}F_{AC} = 0$$
$$R_1^A = -\frac{\sqrt{2}}{2}F_{AC} = -\frac{\sqrt{2}}{2}\left(-\frac{6}{5\sqrt{2}}P\right) = \frac{3}{5}P$$

Thus

$$R_1^A = \frac{3}{5}P$$
$$R_1^B = -\frac{3}{5}P$$

Note: It was not necessary to solve for all of the reaction forces. Only the internal forces on the bars F_{AC} and F_{BC} were required, and these were found using method of joints at C.

Next, applying the constitutive relation

$$\delta_{XY} = \frac{F_{XY}L_{XY}}{E_{XY}A_{XY}}$$

We have

$$\begin{aligned} \delta_{AC} &= \frac{F_{AC}L_{AC}}{E_{AC}A_{AC}} = \frac{F_{AC}L_{AC}}{EA} \\ \delta_{BC} &= \frac{F_{BC}L_{BC}}{E_{BC}A_{BC}} = \frac{F_{BC}L_{BC}}{EA} \end{aligned}$$

By compatibility

$$\begin{aligned} \delta_{AC} &= \mathbf{u}^C \cdot \left(-\frac{1}{\sqrt{2}}\mathbf{e}_1 - \frac{1}{\sqrt{2}}\mathbf{e}_2\right) = -\frac{1}{\sqrt{2}}(u_1^C + u_2^C) \\ \delta_{BC} &= \mathbf{u}^C \cdot \left(\frac{3}{\sqrt{13}}\mathbf{e}_1 - \frac{2}{\sqrt{13}}\mathbf{e}_2\right) = \frac{1}{\sqrt{13}}(3u_1^C - 2u_2^C) \end{aligned}$$

Eliminating u_1^C gives

$$\sqrt{2}\delta_{AC} + \frac{\sqrt{13}}{3}\delta_{BC} = -\frac{5}{3}u_2^C$$

Substituting our variables, and solving for area A and ultimately radius r

$$\begin{aligned} \sqrt{2}\delta_{AC} + \frac{\sqrt{13}}{3}\delta_{BC} &= -\frac{5}{3}u_2^C \\ \sqrt{2}\left(\frac{F_{AC}L_{AC}}{EA}\right) + \frac{\sqrt{13}}{3}\left(\frac{F_{BC}L_{BC}}{EA}\right) &= -\frac{5}{3}u_2^C \\ A = \pi r^2 &= \frac{\sqrt{2}\frac{F_{AC}L_{AC}}{E} + \frac{\sqrt{13}}{3}\frac{F_{BC}L_{BC}}{E}}{\frac{-5}{3}u_2^C} \\ r &= \sqrt{\frac{\sqrt{2}\frac{F_{AC}L_{AC}}{E} + \frac{\sqrt{13}}{3}\frac{F_{BC}L_{BC}}{E}}{\frac{-5}{3}\pi u_2^C}} \end{aligned}$$

We enforce that $u_2^C = 0.01$ m and substitute all values which are known

$$F_{AC} = -\frac{6}{5\sqrt{2}}P = -\frac{6}{5\sqrt{2}}(10000 \text{ N})$$
$$F_{BC} = -\frac{\sqrt{13}}{5}P = -\frac{\sqrt{13}}{5}(10000 \text{ N})$$
$$L_{AC} = 2\sqrt{2} \text{ m}$$
$$L_{BC} = \sqrt{13} \text{ m}$$
$$E = 200 \cdot 10^9 \text{ GPa}$$
$$u_2^C = 0.01 \text{ m}$$

Solving for r yields

$$\boxed{r = 0.0025 \text{ m}}$$

As we will see later in the course, it is possible that the compressive forces will make the bars buckle and this could be the limiting design consideration.

Problems M-4.4
(thermal loads)

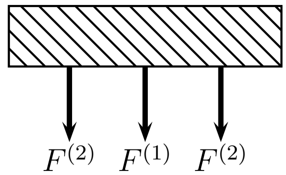
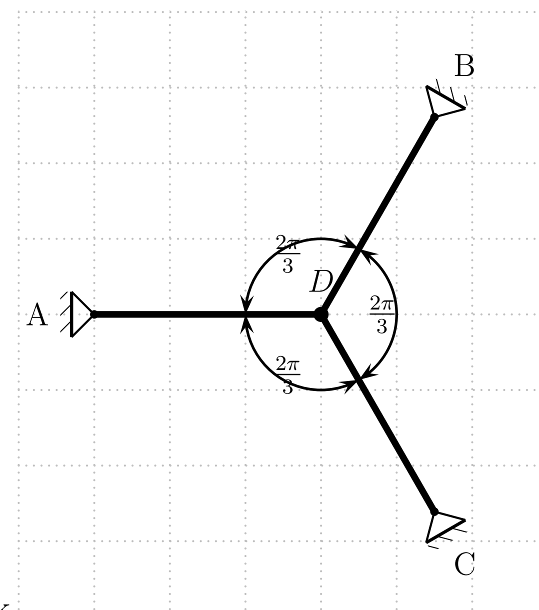


Figure 5: Bar assembly subject to temperature changes

Consider the assembly of bars shown in Figure 5. The center bar is made of a material with Young's modulus $E^{(1)}$, area of the cross section A and coefficient of thermal expansion (CTE) $\alpha^{(1)}$, whereas the respective properties of the left and right bars are $E^{(2)}, A, \alpha^{(2)}$. A rigid block at the top constrans the deformation of the bars but the block is free to move in the vertical direction by an arbitrary value $\bar{\delta}$. However, the block cannot rotate. A temperature change of $\Delta\theta$ is applied after assembly.

(a) (2 points) Express in mathematical form all the relevant principles that apply in this problem both for the individual bars and for the assembly. Make sure you draw Free Body Diagrams as necessary.



Solution: Equilibrium: FBD of the block

$$2F^{(2)} + F^{(1)} = 0$$

Constitutive response of each bar: $\delta^{(i)} = \frac{F^{(i)}L}{E^{(i)}A} + \alpha^{(i)}\Delta\theta L$.
Compatibility condition: The three bars must have the same deflection which must be equal to the deflection of the block, i.e. $\delta^{(i)} = \delta$ for all the bars.

(b) (1 point) Identify all the unknowns of the problem. Count the number of equations you have for each principle and comment on the "solvability" of the problem. Is the problem statically determinate or indeterminate?

Solution:
Unknowns

- Equilibrium: 2 unknowns ($F^{(1)}, F^{(2)}$, (by symmetry we assume the left and right bars have the same load $F^{(2)}$.
- Constitutive: 2 new unknowns $\delta^{(1)}, \delta^{(2)}$.
- Compatibility: 1 additional unknown $\bar{\delta}$. Total: 5 unknowns

Equations

- Equilibrium: only 1 relevant equilibrium equation (sum of forces in the vertical direction equal to zero).
- Constitutive: 2, one for each bar of the form above.
- Compatibility: 2, one for each bar of the form above.

In principle, the problem can be solved as we have as many equations as unknowns. The problem is statically indeterminate as we have two force unknowns and one equilibrium equation.

(c) (3 points) Derive expressions for the forces in the bars and the displacement δ due to the "temperature loading" of the assembly.

Solution: From compatibility $\delta^{(1)} = \delta^{(2)} = \bar{\delta}$, and constitutive:

$$\frac{F^{(1)}\cancel{L}}{E^{(1)}A} + \alpha^{(1)}\Delta\theta\cancel{L} = \frac{F^{(2)}\cancel{L}}{E^{(2)}A} + \alpha^{(2)}\Delta\theta\cancel{L}$$

which can be rewritten as

$$\frac{F^{(2)}}{E^{(2)}} - \frac{F^{(1)}}{E^{(1)}} = (\alpha^{(1)} - \alpha^{(2)})A\Delta T$$

From this equation we learn that if the two materials have the same CTE, the forces on the bars will be zero, as all bars will expand or contract by the same amount regardless of any other properties (E, A, L). Combining with the equilibrium equation:

$$\frac{F^{(2)}}{E^{(2)}} - \frac{\overbrace{F^{(1)}}^{-2F^{(2)}}}{E^{(1)}} = (\alpha^{(1)} - \alpha^{(2)})A\Delta\theta$$

$$F^{(2)}\left(\frac{1}{E^{(2)}} + \frac{2}{E^{(1)}}\right) = (\alpha^{(1)} - \alpha^{(2)})A\Delta\theta$$

$$F^{(2)}\left(\frac{E^{(1)} + 2E^{(2)}}{E^{(1)}E^{(2)}}\right) = (\alpha^{(1)} - \alpha^{(2)})A\Delta\theta$$

$$\boxed{F^{(2)} = \frac{E^{(1)}E^{(2)}}{E^{(1)} + 2E^{(2)}}A(\alpha^{(1)} - \alpha^{(2)})\Delta\theta}$$

$$\boxed{F^{(1)} = -2\frac{E^{(1)}E^{(2)}}{E^{(1)} + 2E^{(2)}}A(\alpha^{(1)} - \alpha^{(2)})\Delta\theta}$$

Replacing the values of the forces in the constitutive and compatibility equations, we find:

$$\bar{\delta} = \delta^{(1)} = \delta^{(2)} = \overbrace{\frac{E^{(1)}\cancel{E^{(2)}}}{E^{(1)} + 2E^{(2)}}\cancel{A}(\alpha^{(1)} - \alpha^{(2)})\Delta\theta}^{F^{(2)}}\,\frac{L}{\cancel{E^{(2)}}\cancel{A}} + \alpha^{(2)}\Delta\theta L$$

$$= \frac{E^{(1)}(\alpha^{(1)} - \cancel{\alpha^{(2)}})\Delta\theta L + \left(\cancel{E^{(1)}} + 2E^{(2)}\right)\alpha^{(2)}\Delta\theta L}{E^{(1)} + 2E^{(2)}}$$

$$\boxed{\bar{\delta} = \left[\frac{E^{(1)}\alpha^{(1)} + 2E^{(2)}\alpha^{(2)}}{E^{(1)} + 2E^{(2)}}\right]\Delta\theta L}$$

There is an interesting interpretation of this result: the structure elongation is as if the whole system had an effective CTE which is the weighted-average of the CTEs of the two materials, where the weights are the stiffnesses of the bars involved.

(d) (2 points) Show that the deflection of the system $\bar{\delta}$ does not depend on the magnitude of the individual material Young's moduli $E^{(1)}, E^{(2)}$ but only on their relative value $\eta = \frac{E^{(1)}}{E^{(2)}}$. Obtain an expression for the deflection in terms of η.

Solution: Replace $E^{(1)} = \eta E^{(2)}$ in the boxed equations above:

$$\bar{\delta} = \frac{\overbrace{\eta E^{(2)}}^{E^{(1)}} \alpha^{(1)} + 2E^{(2)}\alpha^{(2)}}{\underbrace{\eta E^{(2)}}_{E^{(1)}} + 2E^{(2)}} \Delta\theta L$$

$$\bar{\delta} = \frac{\eta\alpha^{(1)} + 2\alpha^{(2)}}{\eta + 2} \Delta\theta L$$

It can be seen that as bar 2 gets much stiffer than 1 (e.g. bar two is made of steel and bar one is made of cheese), $\eta \to 0$, the deflection corresponds to that of the two bars with material 2. Conversely, when bar 1 is much stiffer, $\eta \to \infty$, it dictates the deflection.

○ **Problems M-4.5**

The structure in Figure 6 is composed of three bars with the same cross-sectional area $A = 10\ mm^2$ and length $L = 3\ m$, but of materials with different elastic (Young's) moduli: bar AD has $E^{(1)} = 70$ GPa, bar BD has $E^{(2)} = 120$ GPa, bar CD, has $E^{(3)} = 210$ GPa. The corresponding CTEs are $23, 9\,\text{and}\,13 \times 10^{-6}\text{K}^{-1}$. The structure is subjected to a temperature increase $\Delta\theta = 100\text{K}$.

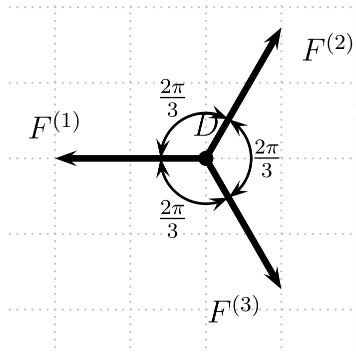
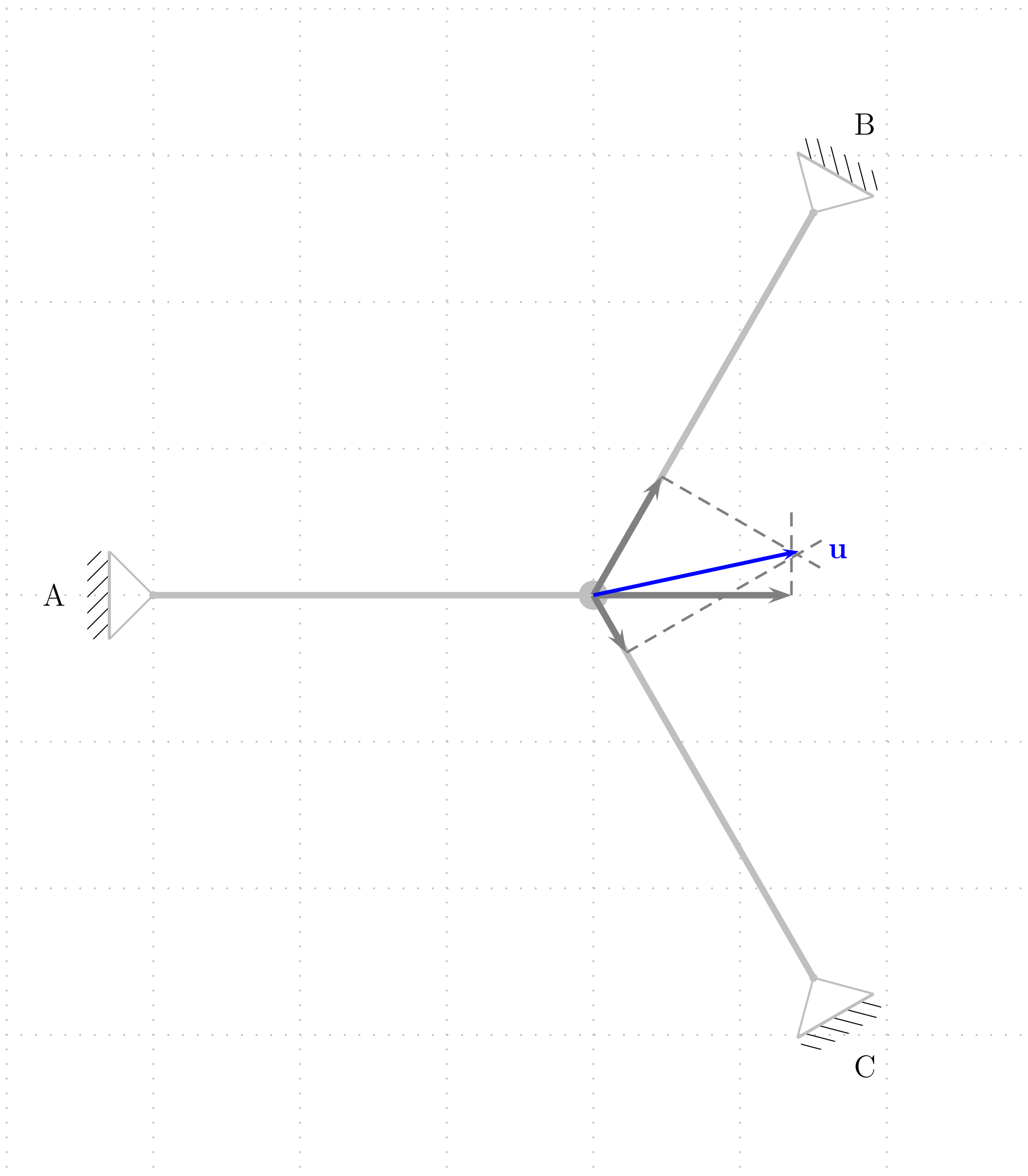


Figure 6: Three bar structure subject to temperature increase

(a) (3 points) Compute the forces in each bar and the deflection of point D caused by the temperature change.

Solution:

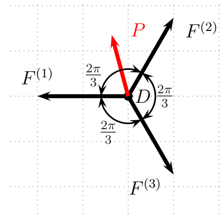
Following the usual script, we first analyze **equilibrium**: Consider the free body diagram of the three bar structure. To reduce clutter, let's call bar AD bar 1, bar BD bar 2 and bar CD bar 3.



This is a planar system of concurrent forces. Applying the two equations of translational **equilibrium** we find that the system is statically indeterminate. Although we cannot find the value of the forces from equilibrium alone, we can conclude from the symmetry of the problem that all three forces must be the same: $F^{(1)} = F^{(2)} = F^{(3)} = F$. (If you are not convinced, think of projecting any two forces in the direction perpendicular to the third, to conclude that those two need to be the same, and, ergo, the third as well).

Now let's consider the **constitutive law** in each bar: $\delta^{(i)} = \frac{FL}{E^{(i)}A} + \alpha^{(i)}\Delta\theta L$.

Compatibility:



Use the fact that the vector $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ projected on the directions of the undeformed bars give their elongation, i.e.: $\mathbf{u}.\mathbf{e}^{(i)} = \delta^{(i)}$ (here (i) is a label of the unit vector of bar (i)):

$$\mathbf{e}^{(1)} = 1\mathbf{e}_1 + 0\mathbf{e}_2,\ \mathbf{e}^{(2)} = -\frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2,\ \mathbf{e}^{(3)} = -\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2$$

$$\mathbf{u}\cdot\mathbf{e}^{(1)} = u_1 = \delta^{(1)}$$

$$\mathbf{u}\cdot\mathbf{e}^{(2)} = -\frac{1}{2}u_1 - \frac{\sqrt{3}}{2}u_2 = \delta^{(2)}$$

$$\mathbf{u}\cdot\mathbf{e}^{(3)} = -\frac{1}{2}u_1 + \frac{\sqrt{3}}{2}u_2 = \delta^{(3)}$$

The three compatibility equations can be combined into one containing only a relation between the three bar elongations by: adding the last two to eliminate u_2, and using the first to replace u_1 with $\delta^{(1)}$. We obtain:

$$-\delta^{(1)} = \delta^{(2)} + \delta^{(3)}$$

$$\text{or } \delta^{(1)} + \delta^{(2)} + \delta^{(3)} = 0$$

which could be anticipated also from the symmetry of the problem. Replacing from the constitutive equations for each bar, we obtain a single equation for F.

$$F\frac{\cancel{L}}{A}\left(\frac{1}{E^{(1)}} + \frac{1}{E^{(2)}} + \frac{1}{E^{(3)}}\right) + \Delta\theta\cancel{L}\left(\alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)}\right) = 0$$

$$\boxed{F = -\Delta\theta A\left(\alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)}\right)\frac{E^{(1)}E^{(2)}E^{(3)}}{E^{(1)}E^{(2)} + E^{(2)}E^{(3)} + E^{(3)}E^{(1)}}}$$

Once F is found, we can go back to the constitutive laws for the bars and find the bar elongations and obtain:

$$\boxed{\delta^{(1)} = \Delta\theta L\left(\alpha^{(1)} - \frac{E^{(2)}E^{(3)}(\alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)})}{E^{(1)}E^{(2)} + E^{(2)}E^{(3)} + E^{(3)}E^{(1)}}\right)}$$

$$\boxed{\delta^{(2)} = \Delta\theta L\left(\alpha^{(2)} - \frac{E^{(3)}E^{(1)}(\alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)})}{E^{(1)}E^{(2)} + E^{(2)}E^{(3)} + E^{(3)}E^{(1)}}\right)}$$

$$\boxed{\delta^{(3)} = \Delta\theta L\left(\alpha^{(3)} - \frac{E^{(1)}E^{(2)}(\alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)})}{E^{(1)}E^{(2)} + E^{(2)}E^{(3)} + E^{(3)}E^{(1)}}\right)}$$

And u_1, u_2 can be obtained from the compatiblity equations:

$$\boxed{u_1 = \delta^{(1)} = \Delta\theta L \left(\alpha^{(1)} - \frac{E^{(2)}E^{(3)}(\alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)})}{E^{(1)}E^{(2)} + E^{(2)}E^{(3)} + E^{(3)}E^{(1)}}\right)}$$

$$u_2 = \frac{\sqrt{3}}{3}\left(2\delta^{(3)} + u_1\right)$$

$$\boxed{u_2 = \frac{\Delta\theta L\left[-E^{(1)}E^{(2)}(\alpha^{(1)} + 2\alpha^{(2)}) + E^{(1)}E^{(3)}(\alpha^{(1)} + 2\alpha^{(3)}) + E^{(2)}E^{(3)}(\alpha^{(3)} - \alpha^{(2)})\right]}{\sqrt{3}\left[E^{(1)}(E^{(2)} + E^{(3)}) + E^{(2)}E^{(3)}\right]}}$$

Replacing all the values of the problem material properties and geometry, we get:

$$\boxed{F \sim -1.643\text{kN}}$$

$$\boxed{u_1 \sim -0.143\text{mm}, u_2 \sim 1.71\text{mm}}$$

$$\boxed{\delta^{(1)} \sim -0.143\text{mm}, \delta^{(2)} \sim -1.41\text{mm}, \delta^{(3)} \sim 1.55\text{mm}}$$

(b) (2 points) Find the load P (magnitude and direction), that needs to be applied at point D to eliminate the displacement produced by the temperature change and return joint D to the original location.

Solution: In this second part, the external load applied at D eliminates the displacement. Therefore, we have $\delta^{(1)} = \delta^{(2)} = \delta^{(3)} = 0$. We consider the constitutive laws in each of the beams:

$$\frac{F^{(i)}L}{E^{(i)}A} = \alpha^{(i)}\Delta\theta L = \delta^{(i)} = 0$$

From the constitutive laws we can find the internal loads in each of the beams. Note that the internal forces are not necessarily equal in magnitude in this second part, because there is an external force P applied to node D.

$$\mathbf{F}^{(i)} = \alpha^{(i)}\Delta\theta E^{(i)}A\mathbf{e}^{(i)}$$

Inserting this expression in the equilibrium of forces at D, we get:

$$-\mathbf{F}^{(1)} - \mathbf{F}^{(2)} - \mathbf{F}^{(3)} + \mathbf{P} = 0$$

Projecting this equation in the x and y directions, we get:

$$\sum F_1 = -\alpha^{(1)}\Delta\theta E^{(1)}A + \frac{1}{2}\alpha^{(2)}\Delta\theta E^{(2)}A + \frac{1}{2}\alpha^{(3)}\Delta\theta E^{(3)}A + P_1 = 0$$

$$\sum F_2 = \frac{\sqrt{3}}{2}\alpha^{(2)}\Delta\theta E^{(2)}A - \frac{\sqrt{3}}{2}\alpha^{(3)}\Delta\theta E^{(3)}A + P_2 = 0$$

The final form for **P** in each direction is given here:

$$P_1 = \Delta\theta A \left[\alpha^{(1)}E^{(1)} - \frac{1}{2}(\alpha^{(2)}E^{(2)} + \alpha^{(3)}E^{(3)})\right] = -295N$$

$$P_2 = \frac{\sqrt{3}}{2}\Delta\theta A \left[\alpha^{(3)}E^{(3)} - \alpha^{(2)}E^{(2)}\right] = 1429N$$

The magnitude of the external force **P** is therefore:

$$\boxed{|\mathbf{P}| \sim -1460\text{N}}$$

The equilibrium at node D with the external load applied **P** is shown in the figure below.