16.001 - Materials & Structures
Problem Set #3

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THIS PSET ONLY HAS TWO PROBLEMS AT THE END FOR GRADE, THE REST OF THE PROBLEMS ARE GIVEN WITH SOLUTIONS AS A LEARNING EXERCISE

○ **Problem M-3.1**
(MO: M6)

Consider the pin-jointed cantilever truss shown in Figure 1.

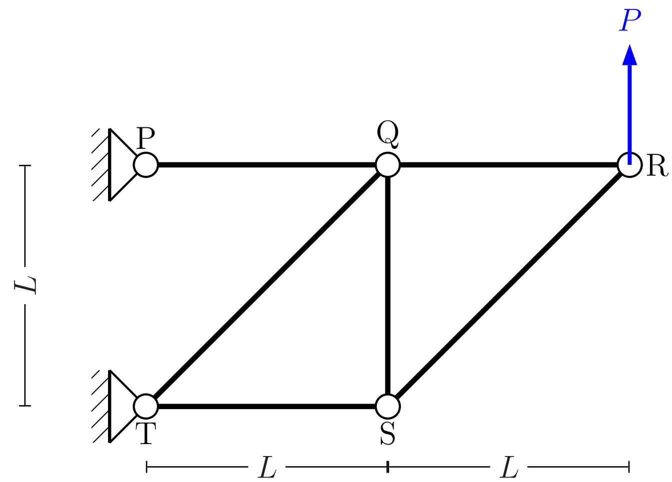
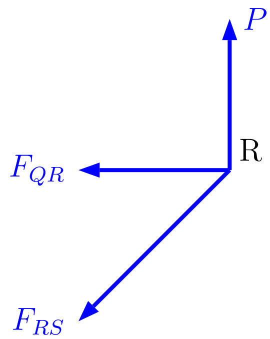


Figure 1: Cantilever Truss

(a) Determine the forces all of the members of the truss due to the applied load P, distinguishing between tension and compression.

Solution: We may use the method of joints to solve for the forces in the members of the truss. Beginning with summation of forces in Joint R,

- **Joint R**



  – Force equilibrium in $\mathbf{e}_2$ direction

$$\sum F_2 = 0$$

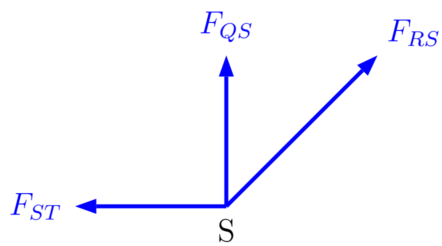
$$P - F_{RS}\frac{\sqrt{2}}{2} = 0$$

$$\boxed{F_{RS} = \sqrt{2}P}$$

– Force equilibrium in $\mathbf{e}_1$ direction

$$\sum F_1 = 0$$
$$-F_{QR} - \frac{\sqrt{2}}{2} F_{RS} = 0$$
$$F_{QR} = -\frac{\sqrt{2}}{2} F_{RS} = -\frac{\sqrt{2}}{2}(\sqrt{2}P)$$
$$\boxed{F_{QR} = -P}$$

- **Joint S**



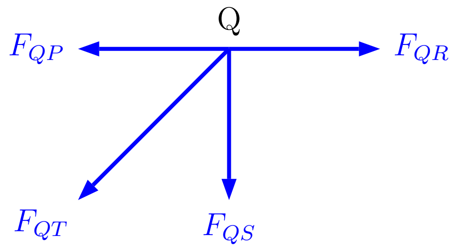
– Force equilibrium in $\mathbf{e}_1$ direction

$$\sum F_1 = 0$$
$$F_{RS}\frac{\sqrt{2}}{2} - F_{ST} = 0$$
$$F_{ST} = F_{RS}\frac{\sqrt{2}}{2} = (\sqrt{2}P)\left(\frac{\sqrt{2}}{2}\right)$$
$$\boxed{F_{ST} = P}$$

– Force equilibrium in $\mathbf{e}_2$ direction

$$\sum F_2 = 0$$
$$F_{QS} + F_{RS}\frac{\sqrt{2}}{2} = 0$$
$$F_{QS} = -F_{RS}\frac{\sqrt{2}}{2} = -(\sqrt{2}P)(\frac{\sqrt{2}}{2})$$
$$\boxed{F_{QS} = -P}$$

- **Joint Q**



– Force equilibrium in $\mathbf{e}_2$ direction

$$\sum F_2 = 0$$
$$-F_{QT}\frac{\sqrt{2}}{2} - F_{QS} = 0$$
$$F_{QT} = -\sqrt{2}F_{QS} = -\sqrt{2}(-P)$$
$$\boxed{F_{QT} = \sqrt{2}P}$$

– Force equilibrium in $\mathbf{e}_1$ direction

$$\sum F_1 = 0$$
$$-F_{PQ} - F_{QT}\frac{\sqrt{2}}{2} + F_{QR} = 0$$
$$F_{PQ} = -F_{QT}\frac{\sqrt{2}}{2} + F_{QR} = -(-\sqrt{2}P)(\frac{\sqrt{2}}{2}) + (-P)$$
$$\boxed{F_{PQ} = -2P}$$

Using the method of joints and analyzing joints R, S, and Q gives the forces in the 6 members of the truss. In summary, we have that the forces in the rods are:

$\boxed{F_{RS} = \sqrt{2}P}$ tensile

$\boxed{F_{QR} = P}$ compressive

$\boxed{F_{ST} = P}$ tensile

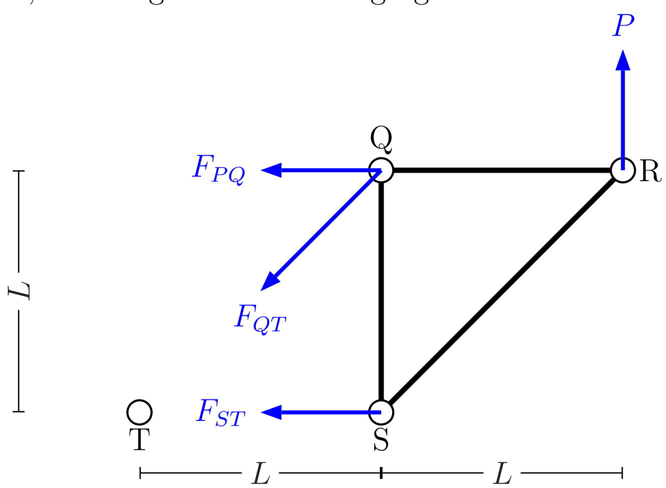
$\boxed{F_{QS} = P}$ compressive

$\boxed{F_{QT} = \sqrt{2}P}$ tensile

$\boxed{F_{PQ} = 2P}$ compressive

(b) Assume now that we were only interested in the force in member PQ. What method could you use to easily find the force in PQ (without knowledge of the internal forces in the other members)? Perform the method and obtain this force.

Solution: We could apply the method of sections in order to determine the internal force in member PQ. Specifically, we may take a cut through members PQ, QT and ST, resulting in the following figure:



Taking moments about the point T (not connected to the newly cut structure, but shown for reference), we obtain the following equation

$$\sum M_T = 0 \rightarrow (2L)(P) + (L)(F_{PQ}) = 0 \quad (1)$$

Solving for F_{PQ} gives $F_{PQ} = -2P$, which agrees with the value found in part (a).

(c) What is the maximum load P the truss may support if the members can only handle a maximum tension or compression of 20 kN?

Solution: To solve this problem, we apply linearity and realize that the forces experienced by individual members of the truss scale with the applied force P. The maximum of these tensile/compressive forces is experienced by rod PQ which experiences a compressive force of $2P$ due to the applied load of P at joint R. Thus

$$2P = 20\ kN$$

$$\boxed{P = 10kN}$$

Thus, the maximum allowable load P such that no member of the truss experiences a force of more than 20 kN in magnitude is $\boxed{10\ kN}$.

○ **Problem M-3.2**

A space truss is a three-dimensional truss structure, an example of which is given in Figure 2. The four ball joints are attached to a fixed wall. The three little bars attached to the ball joints can rotate freely. The forces applied on B and C have a magnitude of 2kN each.

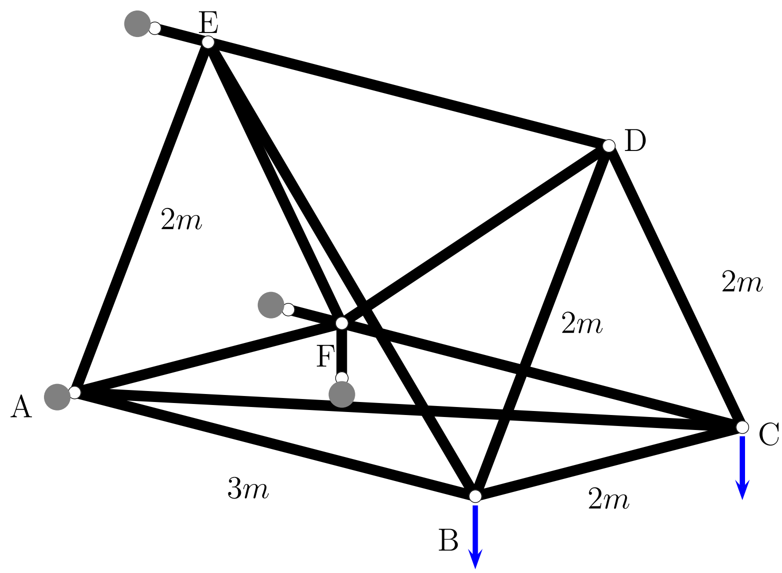
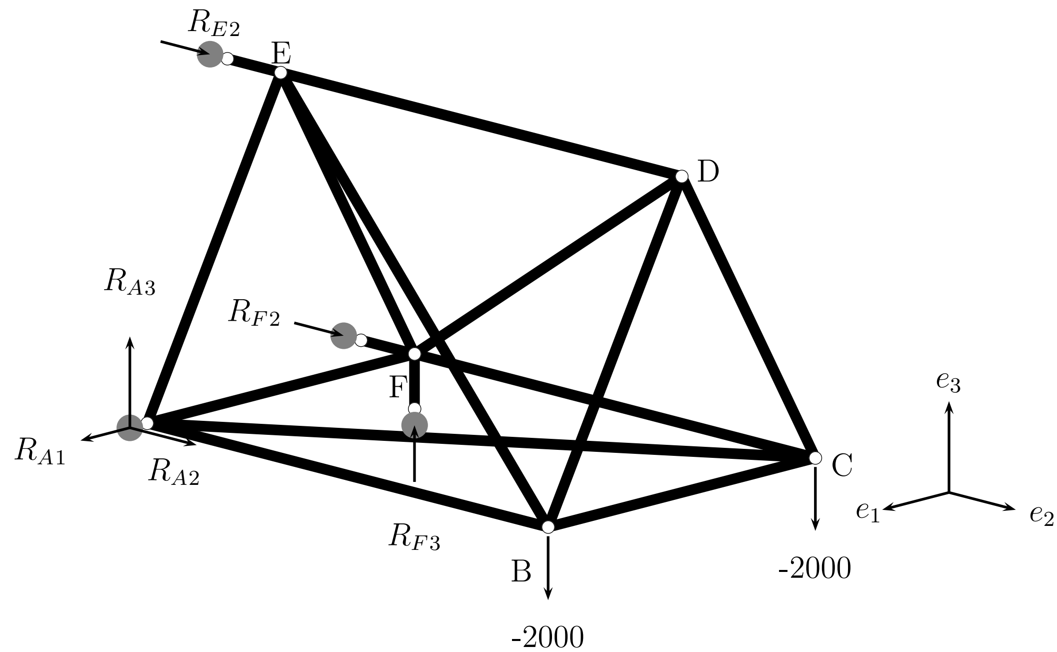


Figure 2: Space truss

(a) Draw a FBD of this problem and determine the reaction at the supports. Is the system statically determinate?

Solution: We begin by establishing our coordinate system. Select F to be the origin, $\mathbf{e_1}$ to align with AF, $\mathbf{e_2}$ to align with FC, and $\mathbf{e_3}$ to point upwards.



$$\sum F_1 = 0 \rightarrow R_{A_1} = 0 \tag{2}$$

$$\sum F_2 = 0 \rightarrow R_{A_2} + R_{F_2} + R_{E_2} = 0 \tag{3}$$

$$\sum F_3 = 0 \rightarrow R_{A_3} + R_{F_3} - 4 = 0 \tag{4}$$

Take moments around point A,

$$\sum \mathbf{M}_A = 0 = \mathbf{r_B} \times \mathbf{P} + \mathbf{r_C} \times \mathbf{P} + \mathbf{r_F} \times \mathbf{R_{F_3}} + \mathbf{r_F} \times \mathbf{R_{F_2}} + \mathbf{r_E} \times \mathbf{R_{E_2}}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$+ \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ R_{F_3} \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ R_{F_2} \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ \sqrt{3} \end{bmatrix} \times \begin{bmatrix} 0 \\ R_{E_2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -6 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2R_{F_3} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2R_{F_2} \end{bmatrix} + \begin{bmatrix} -\sqrt{3}R_{E_2} \\ 0 \\ -R_{E_2} \end{bmatrix}$$

$$\sum M_1 = 0 \rightarrow -12 - \sqrt{3}R_{E_2} = 0$$
$$\sum M_2 = 0 \rightarrow -4 + 2R_{F_3} = 0$$
$$\sum M_3 = 0 \rightarrow -2R_{F_2} - R_{E_2} = 0$$

Therefore

$$R_{E_2} = -4\sqrt{3}\text{kN} \tag{5}$$
$$R_{F_3} = 2\text{kN} \tag{6}$$
$$R_{F_2} = -0.5R_{E_2} = 2\sqrt{3}\text{kN} \tag{7}$$

Plug Eq.7 in Eq.4, we have

$$R_{A_1} = 0$$
$$R_{A_2} = -R_{E_2} - R_{F_2} = 2\sqrt{3}\text{kN}$$
$$R_{A_3} = 4 - R_{F_3} = 2\text{kN}$$

(b) Determine the force in members AB, CD, ED, and CF.

Solution: There are several approaches to solve this problem. In this solution, we apply method of joints directly to joints B, C, and D, and solve for the internal forces in the truss members.

Now we apply method of joints to point C, which connects members BC, CD, and CF. In 3D, equilibrium gives us 3 equations at every joint. The force equilibrium equation at joint C results in the following equations:

- **Equilibrium at Joint A**

$$\sum F_1 = 0 \rightarrow R_{A_1} - \frac{1}{2}F_{AE} - F_{AF} - \frac{2}{\sqrt{13}}F_{AC} = 0 \tag{8}$$
$$\sum F_2 = 0 \rightarrow R_{A_2} + F_{AB} + \frac{3}{\sqrt{13}}F_{AC} = 0 \tag{9}$$
$$\sum F_3 = 0 \rightarrow \frac{\sqrt{3}}{2}F_{AE} + R_{A_3} = 0 \tag{10}$$

Therefore we have,

$$F_{AE} = \frac{-4}{\sqrt{3}}\text{kN}$$

- **Equilibrium at Joint E**

$$\sum F_1 = 0 \rightarrow \frac{1}{2}F_{AE} - \frac{1}{2}F_{EF} + \frac{1}{\sqrt{13}}F_{EB} = 0$$
$$\sum F_2 = 0 \rightarrow F_{ED} + \frac{3}{\sqrt{13}}F_{EB} + R_{E_2} = 0$$
$$\sum F_3 = 0 \rightarrow -\frac{\sqrt{3}}{2}F_{AE} - \frac{\sqrt{3}}{2}F_{EF} - \frac{\sqrt{3}}{\sqrt{13}}F_{EB} = 0$$

Plug in F_{AE}, we have,

$$F_{EB} = \frac{2\sqrt{13}}{\sqrt{3}}\text{kN}$$
$$\boxed{F_{ED} = 2\sqrt{3}}$$
$$F_{EF} = 0$$

- **Equilibrium at Joint B**

$$\sum F_1 = 0 \rightarrow -\frac{1}{2}F_{BD} - F_{BC} - \frac{1}{\sqrt{13}}F_{BE} = 0$$
$$\sum F_2 = 0 \rightarrow -F_{AB} - \frac{3}{\sqrt{13}}F_{BE} = 0$$
$$\sum F_3 = 0 \rightarrow \frac{\sqrt{3}}{2}F_{BD} + \frac{\sqrt{3}}{\sqrt{13}}F_{BE} - 2 = 0$$

Plug in F_{EB}, we have

$$F_{BD} = 0$$
$$\boxed{F_{AB} = -2\sqrt{3}\text{kN}}$$
$$F_{BC} = \frac{-2}{\sqrt{3}}\text{kN}$$

- **Equilibrium at Joint C**

$$\sum F_1 = 0 \rightarrow F_{BC} + \frac{1}{2}F_{CD} + \frac{2}{\sqrt{13}}F_{AC} = 0$$
$$\sum F_2 = 0 \rightarrow -F_{CF} - \frac{3}{\sqrt{13}}F_{AC} = 0$$
$$\sum F_3 = 0 \rightarrow \frac{\sqrt{3}}{2}F_{CD} - 2 = 0$$

Therefore,

$$\boxed{F_{CD} = \frac{4}{\sqrt{3}}}$$

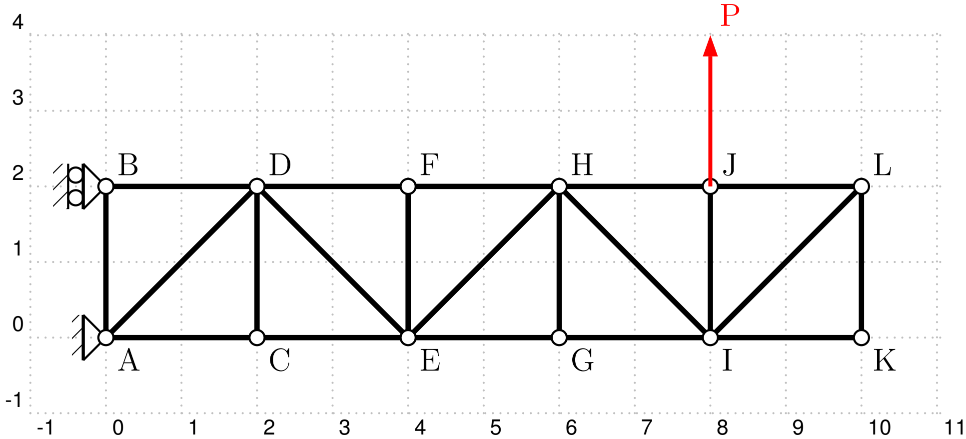
Plug in $F_{BC} = \frac{-2}{\sqrt{3}}$, we have

$$F_{AC} = 0\text{kN}$$

Finally we have

$$\boxed{F_{CF} = -\frac{3}{\sqrt{13}}F_{AC} = 0\text{kN}}$$

Problem M-3.3
For the truss structure in the figure:



(a) (9 points) Can you identify any bars in the structure which carry no internal load for the external load given? Justify your answer.

Solution: AB, CD, EF, GH, KL, JL, IL, IK, HJ

(b) (1 point) Are there any other bars for which the internal load can be inferred without any calculation? Justify your answer.

Solution: Clearly the load in bar IJ is P (tensile).

(c) (3 points) Are there any other bars which you know must carry the same internal load without doing any calculation? Justify your answer.

Solution: $F^{AC} = F^{CE}, F^{DF} = F^{FH}, F^{EG} = F^{GI}$

(d) (3 points) Are the horizontal bars at the top in tension or compression? Compute their internal loads in sequence from the right to the left (this will help you quickly find their values).

Solution: All in compression, except for HJ and JL which have no load. To compute F^{FH}, cut the bar, expose the internal load, take moments wrt E:

$$F^{FH} \cdot 2\text{m} + P \cdot 4\text{m} = 0, \rightarrow \boxed{F^{FH} = -2P = F^{FD}}$$

To compute F^{BD}, cut the bar, expose the internal load, take moments wrt A:

$$F^{BD} \cdot 2\text{m} + P \cdot 8\text{m} = 0, \rightarrow \boxed{F^{BD} = -4P}$$

(e) (4 points) Conduct the same analysis for all the horizontal bars at the bottom. Are they in tension or compression?

Solution: All in tension, except for IK which has no load. To compute F^{GI}, cut the bar, expose the internal load, take moments wrt H:

$$-F^{GI} \cdot 2\text{m} + P \cdot 2\text{m} = 0, \rightarrow \boxed{F^{GI} = P = F^{EG}}$$

To compute F^{CE}, cut the bar, expose the internal load, take moments wrt D:

$$-F^{CE} \cdot 2\text{m} - P \cdot 6\text{m} = 0, \rightarrow \boxed{F^{CE} = 3P = F^{AC}}$$

(f) (4 points) Conduct this analysis one more time for all the diagonal bars.

Solution: Cutting through with a vertical line between HJ HI and GI, drawing the FBD, exposing F^{HI} and doing sum of forces in the vertical direction:

$$F^{HI}\frac{\sqrt{2}}{2} + P = 0, \rightarrow \boxed{F^{HI} = -\sqrt{2}P}$$

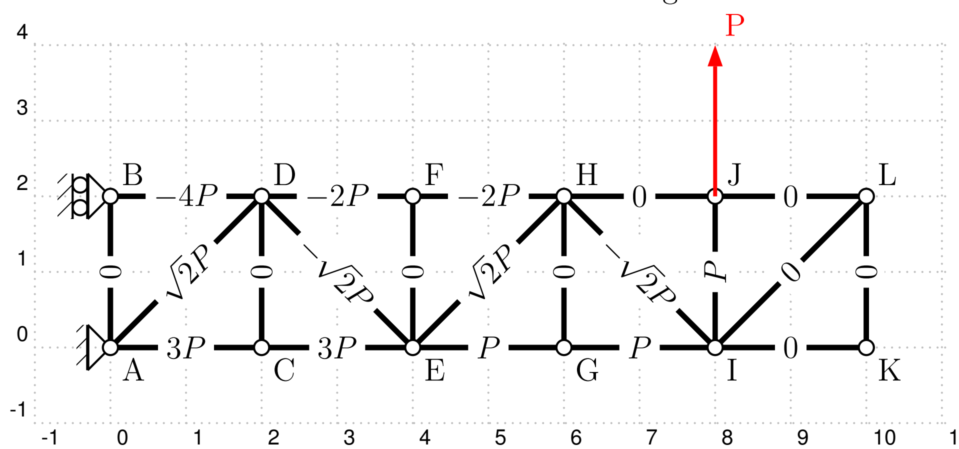
Doing the same throug FH, EH, EG:

$$-F^{EH}\frac{\sqrt{2}}{2} + P = 0, \rightarrow \boxed{F^{EH} = \sqrt{2}P}$$

Similarly, find:

$$\boxed{F^{ED} = -\sqrt{2}P}, \boxed{F^{AD} = \sqrt{2}P}$$

All the internal loads in the bars are shown in the figure:



○ **Problem M-3.4**

Consider the truss shown in Figure 3. The structure is subject to a load with a magnitude $\bar{F} = 600\,\text{N}$ at joint C.

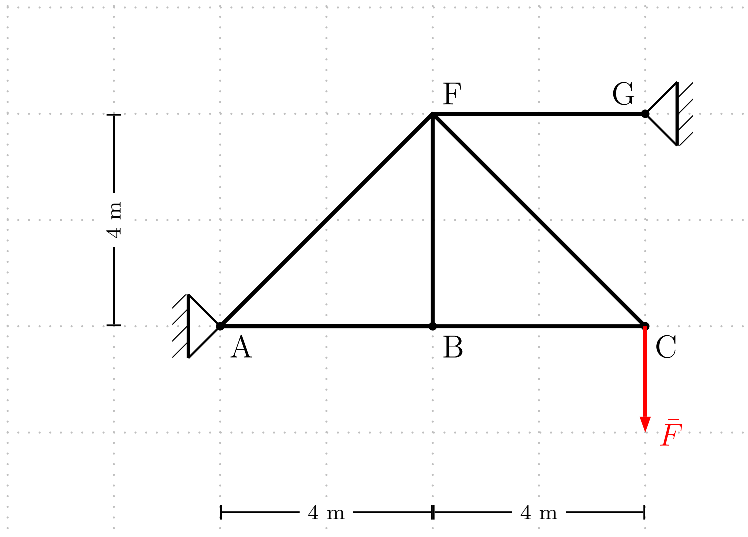


Figure 3: Analysis of a truss structure.

(a) (1 point) Is the system SD, SI or unstable. Justify your answer.

Solution: SD, although two pins, which could be four independent reaction components, the one at point G has a known direction (the direction of bar FG), and therefore constitutes a single unknown (the scalar magnitude)

(b) (3 points) Compute the reactions at the supports.

Solution: To compute the reactions, start by drawing a free body diagram, see Figure 4. From this write the equations of equilibrium.

$$\sum F_1 = 0 = R_{G_1} + R_{A_1}$$
$$\sum F_2 = 0 = R_{A_2} - F$$
$$\sum M_A = 0 = -4R_{G_1} - 8F \rightarrow R_{G_1} = -2F$$
$$\rightarrow R_{A_1} = 2F \rightarrow R_{A_2} = F$$

This results in final values of the reaction:

$$R_{A_1} = 2F = 1200N$$
$$R_{A_2} = F = 600N$$
$$R_{G_1} = -2F = -1200N$$

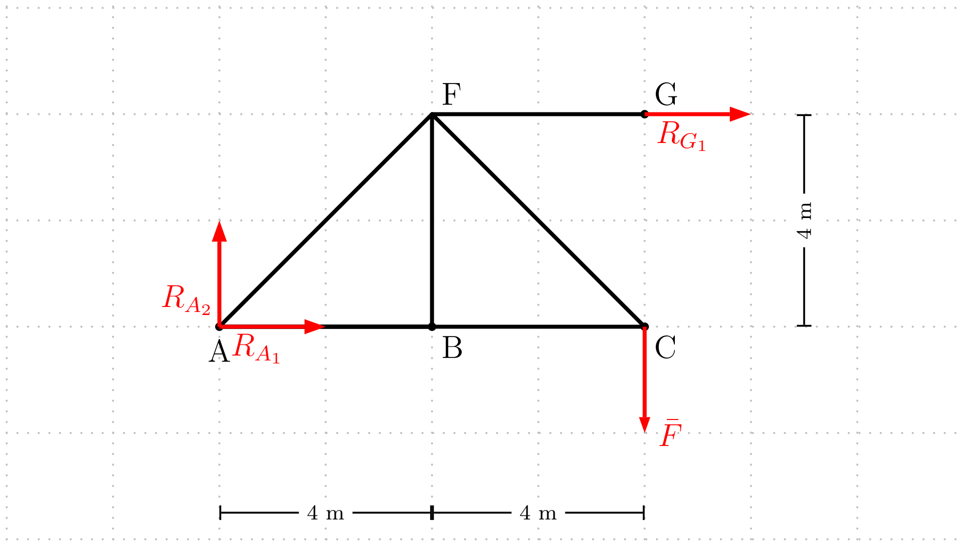


Figure 4

(c) (5 points) Determine the axial forces in the members of the truss.

Solution: Clearly the load on bar BF is zero: at **Joint B** the equation of equilibrium is:

$$\sum F_2 = 0 = F_{BF}$$

and load on AB is equal to that on BC. To find load on BC, use MoM, cut bars FG, FC, BC, keep the left, take moments wrt F:

$$0 = -4m \times F - 4m \times F_{BC}, \rightarrow F_{BC} = F_{AB} = -F = -600N$$

To obtain the load F_{FC}, consider MoJ at **Joint C**. The relevant equation of equilibrium is:

$$\sum F_2 = 0 = -F + \frac{4}{\sqrt{4^2 + 4^2}} F_{FC}, \rightarrow F_{FC} = \sqrt{2}F = 848.5N$$

To obtain the load F_{AF}, consider MoM cutting the truss through AF, AB, keep the left part, take moments wrt B:

$$0 = -4m \times R_{A_2} - \frac{\sqrt{2}}{2} 4m \times F_{AF} = 0, \rightarrow F_{AF} = -\sqrt{2}R_{A_2} = -\sqrt{2}F = -848.5N$$

(d) (5 points) The support G can be moved at your discretion while preserving the length of bar FG, that is, rotating bar FG counterclockwise by an angle α around joint F. Find the optimal angle α_{opt} for the location of support G such that the

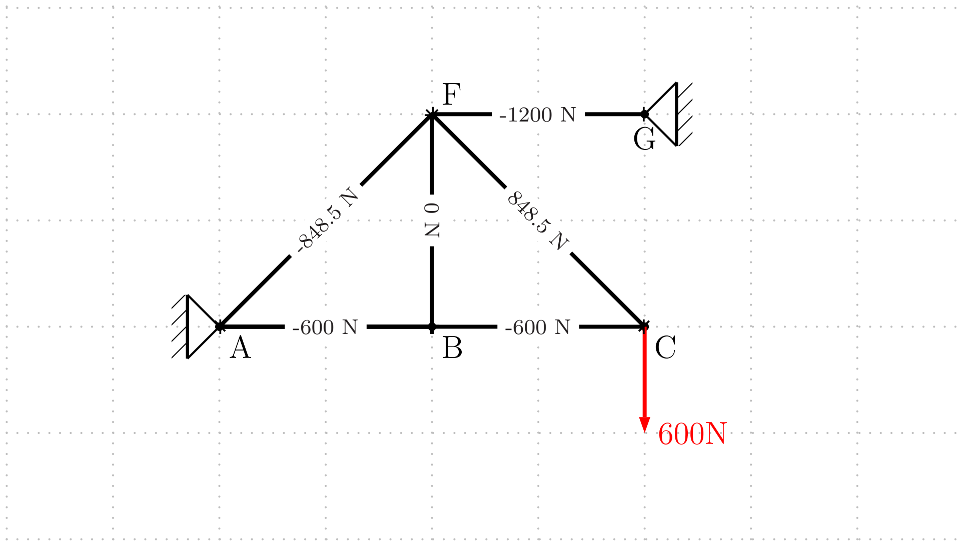
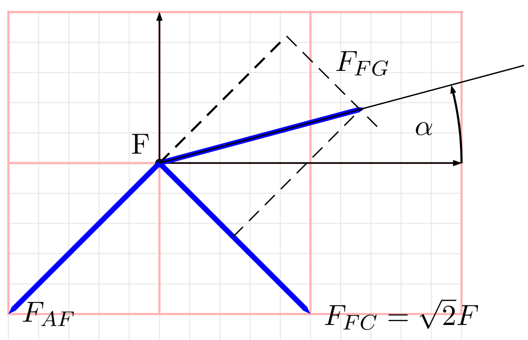


Figure 5

maximum load magnitude in all of the bars is minimized. Is there an angle α_{bad} for which you find a problem with the structure? What problem is it and how do you explain it?

Solution: Start with a general orientation of the bar FG at an angle α wrt the horizontal. Note that F_{FC}, F_{AB}, F_{BC} don't change. Do MoJ at F:



Note that the analysis is much more easily done in the rotated access aligned with bars AF and FC. The two equations of sums of force components along these directions are:

$$F_{FC} + F_{FG}\cos{(\alpha + \frac{\pi}{4})} = 0, \rightarrow \boxed{F_{FG} = \frac{-\sqrt{2}F}{\cos{(\alpha + \frac{\pi}{4})}}}$$

$$F_{FG}\sin{(\alpha + \frac{\pi}{4})} - F_{AF} = 0, \boxed{F_{AF} = -\sqrt{2}\tan{(\alpha + \frac{\pi}{4})}F}$$

To find the extrema, take derivative of either one wrt α, set to zero, solve for α

to find stationary points:

$$0 = \frac{dF_{FG}}{d\alpha} = -\sqrt{F}(-1)\cos^{-2}(\alpha + \frac{\pi}{4})\sin(\alpha + \frac{\pi}{4})$$
$$\rightarrow \sin(\alpha + \frac{\pi}{4}) = 0, \rightarrow \boxed{\alpha_{opt} = \frac{3}{4}\pi, -\frac{1}{4}\pi}$$

For the first solution, the FG is aligned with and opposite to FC:

$$\boxed{F_{FG} = \frac{-\sqrt{2}F}{\cos(\pi)} = \sqrt{2}F, \text{ tensile}}$$
$$\boxed{F_{AF} = -\sqrt{2}\tan(\pi)F = 0}$$

For the second solution, the FG is aligned with and overlaps FC:

$$\boxed{F_{FG} = \frac{-\sqrt{2}F}{\cos(0)} = -\sqrt{2}F, \text{ compressive}}$$
$$\boxed{F_{AF} = -\sqrt{2}\tan(0)F = 0}$$

At any other angle, the load is larger in magnitude in FG (as its projection on FC needs to balance the force on that bar). When the angle wrt FC approaches $\pi/2$, i.e. $\alpha = \pi/4$, FG loses its ability to take a force component parallel to FC and the load grows to $-\infty$. This is $\boxed{\alpha_{bad} = \pi/4}$

$$F_{FG} = \frac{-\sqrt{2}F}{\cos(\frac{\pi}{2}^-)} \rightarrow -\infty$$
$$F_{AF} = -\sqrt{2}\tan(\frac{\pi}{2}^-)F \rightarrow -\infty$$

The moment you cross $\alpha = \pi/4$, the loads start decreasing from their limit values

$$F_{FG} = \frac{-\sqrt{2}F}{\cos(\frac{\pi}{2}^+)} \rightarrow \infty$$
$$F_{AF} = -\sqrt{2}\tan(\frac{\pi}{2}^+)F \rightarrow \infty$$

with a sign change as α continues to increase.

○ **Problem M-3.5**
(MO: M5, M6) A Fink roof truss structure is shown in Figure 6.

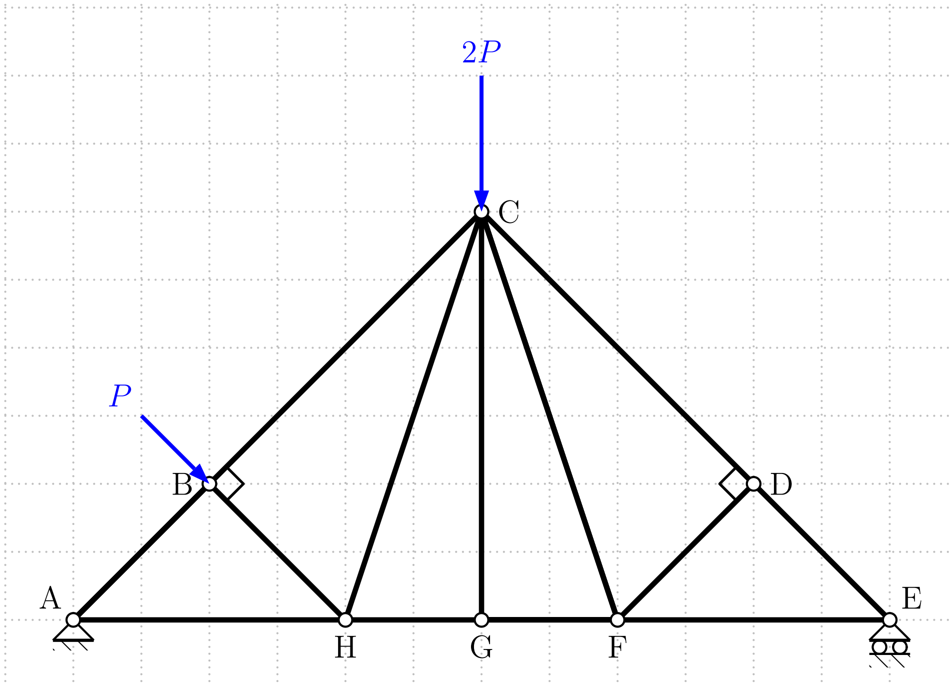
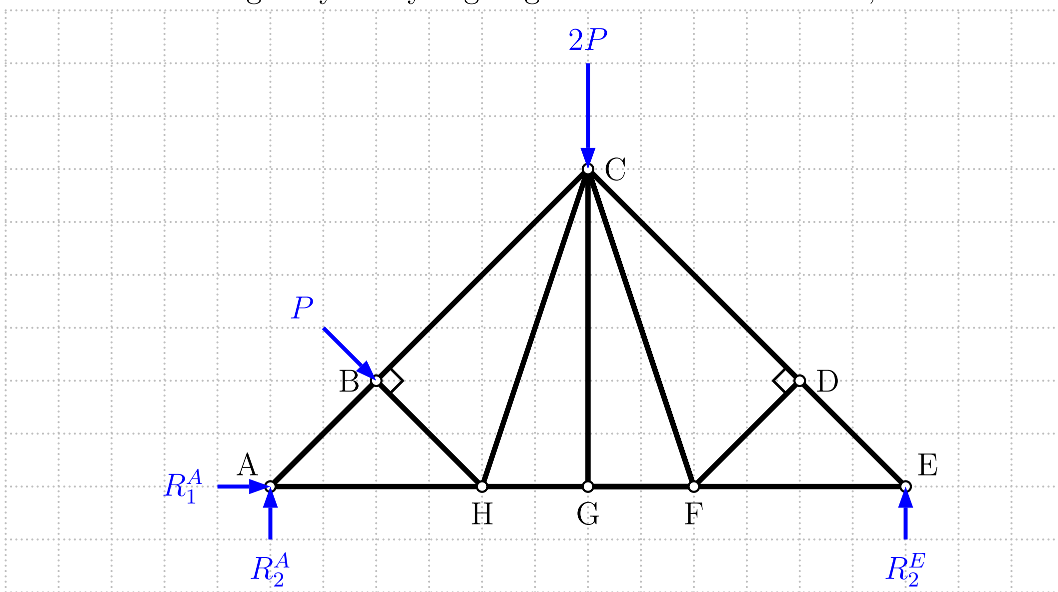


Figure 6: Fink Truss

(a) (5 points) Find the reaction forces at points A and E.

Solution: We begin by analyzing a global FBD of our truss, shown below.



The reaction forces on the truss are R_1^A, R_2^A and R_2^E. We apply force and moment equilibrium to solve for these reactions, as shown below.

- Force equilibrium in 1-direction

$$\sum F_1 = 0$$

$$R_1^A + \frac{\sqrt{2}}{2}P = 0 \rightarrow \boxed{R_1^A = -\frac{\sqrt{2}}{2}P}$$

- Force equilibrium in 2-direction

$$\sum F_2 = 0$$

$$R_2^A + R_2^E - 2P - \frac{\sqrt{2}}{2}P = 0$$

- Moment equilibrium about point A (to eliminate R_1^A, R_2^A)

$$\sum M^A = 0$$

$$-2\sqrt{2}(P) - 6(2P) + 12(R_2^E) = 0$$

$$12(R_2^E) = 12P + 2\sqrt{2}(P)$$

$$\boxed{R_2^E = \left(1 + \frac{\sqrt{2}}{6}\right) P}$$

- Back to force equilibrium in 2-direction

$$R_2^A + R_2^E - 2P - \frac{\sqrt{2}}{2}P = 0$$

$$R_2^A = \left(2 + \frac{\sqrt{2}}{2}\right) P - R_2^E = \left(2 + \frac{\sqrt{2}}{2}\right) P - \left(1 + \frac{\sqrt{2}}{6}\right) P$$

$$\boxed{R_2^A = \left(1 + \frac{\sqrt{2}}{3}\right) P}$$

Thus, we find that

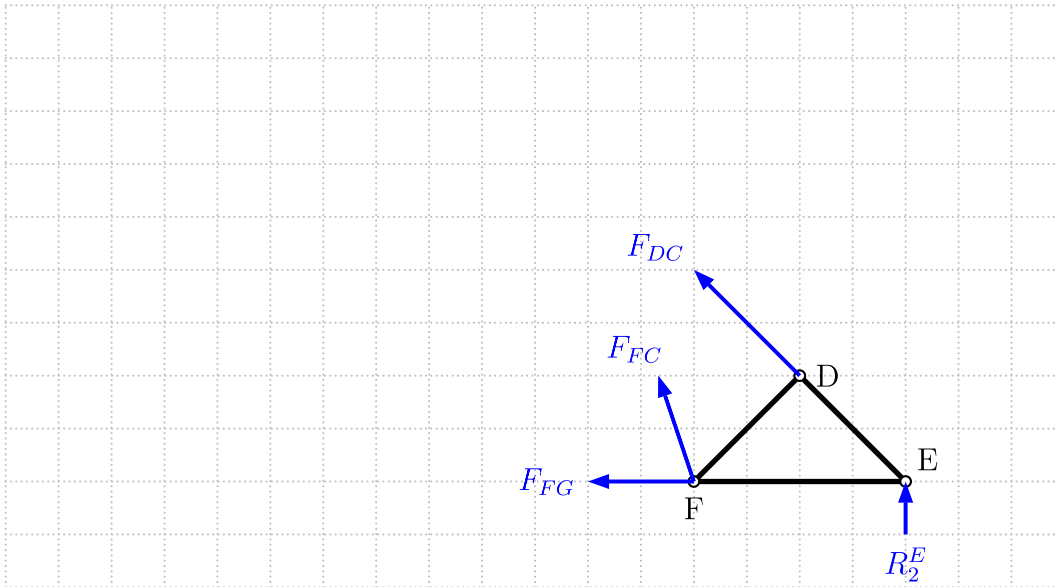
$$R_1^A = -\frac{\sqrt{2}}{2}P$$

$$R_2^A = \left(1 + \frac{\sqrt{2}}{3}\right) P$$

$$R_2^E = \left(1 + \frac{\sqrt{2}}{6}\right) P$$

(b) (5 points) Determine the internal force in bar DC using the method of sections.

Solution: Applying method of sections and taking a cut through the suggested members yields (isolating the right section of the truss)



We may take moments about point F to eliminate all other unknowns besides F_{CD}. The resulting moment equation is:

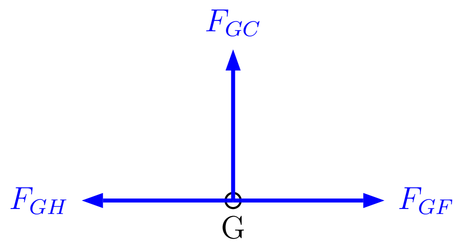
$$\sum M^F = 0$$
$$2\sqrt{2}(F_{CD}) + 4(R_2^E) = 0$$
$$F_{CD} = -\frac{4R_2^E}{2\sqrt{2}} = -\sqrt{2}R_2^E$$
$$F_{CD} = -\sqrt{2}\left(1 + \frac{\sqrt{2}}{6}\right)P = -\left(\sqrt{2} + \frac{1}{3}\right)P$$

Thus, the internal force in bar DC is $\boxed{F_{DC} = -\left(\sqrt{2} + \frac{1}{3}\right)P}$, so it experiences a compressive force.

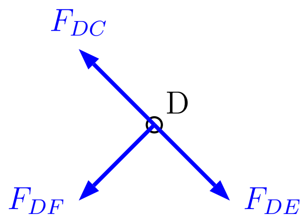
(c) (5 points) Determine which bars in the structure carry no internal load for the external loads given.

Solution: To determine which bars in the structure carry no internal loads, we must enforce equilibrium at every joint. Analyzing joints G, D and F in particular, we find

- Joint G



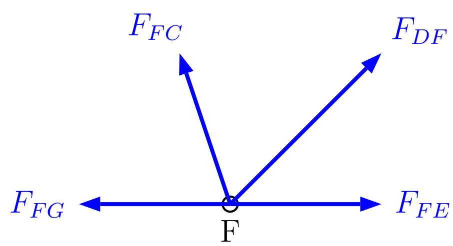
By force equilibrium in the 2-direction, we find $\boxed{F_{FG} = 0}$



- Joint D

By force equilibrium in the direction along F_{DF}, we find that there is no force to counteract F_{DF}, and so $\boxed{F_{DF} = 0}$

- Joint F



By force equilibrium in the 2-direction, and knowing that $F_{DF} = 0$, we find $\boxed{F_{FC} = 0}$

Thus, we find that bars GC, DF, and FC carry no internal loads.