16.001 Unified Engineering
Materials and Structures

Transformation of stress components, principal and maximum shear stresses, Mohr's circle

Reading assignments: CDL 4.2-4.7

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Outline

Transformation of stress components I

Consider a different system of cartesian coordinates $\tilde{\mathbf{e}}_i$. We can express our tensor in either one:

$$\boldsymbol{\sigma} = \sigma_{kl}\mathbf{e}_k \otimes \mathbf{e}_l = \tilde{\sigma}_{mn}\tilde{\mathbf{e}}_m \otimes \tilde{\mathbf{e}}_n \tag{1}$$

We would like to relate the stress components in the two systems. To this end, we take the scalar product of (1) with $\tilde{\mathbf{e}}_i$ and $\tilde{\mathbf{e}}_j$:

$$\tilde{\mathbf{e}}_i \cdot \boldsymbol{\sigma} \cdot \tilde{\mathbf{e}}_j = \sigma_{kl}\left(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_k\right)\left(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_j\right) = \tilde{\sigma}_{mn}\left(\tilde{\mathbf{e}}_i \cdot \tilde{\mathbf{e}}_m\right)\left(\tilde{\mathbf{e}}_n \cdot \tilde{\mathbf{e}}_j\right) = \tilde{\sigma}_{mn}\delta_{im}\delta_{nj} = \tilde{\sigma}_{ij}$$

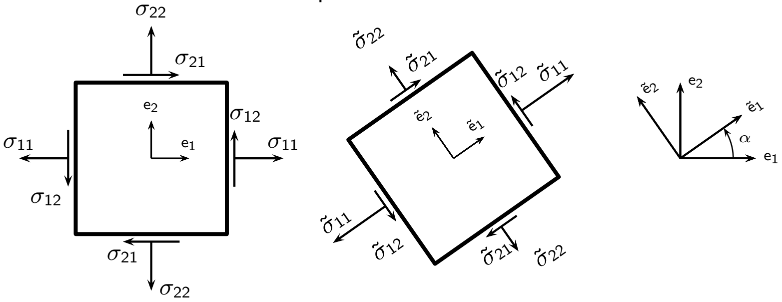
or

Definition

$$\tilde{\sigma}_{ij} = \sigma_{kl}\left(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_k\right)\left(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_j\right) \tag{2}$$

The factors in parenthesis are the cosine directors of the angles between the original and primed coordinate axes: $\left(\tilde{\mathbf{e}}_i \cdot \mathbf{e}_k\right) = \cos\left(\sphericalangle \tilde{\mathbf{e}}_i \mathbf{e}_k\right)$

Transformation of stress components in two dimensional states of stress:



$$\tilde{\sigma}_{\alpha\beta} = \sigma_{\gamma\delta}\left(\mathbf{e}_\gamma \cdot \tilde{\mathbf{e}}_\alpha\right)\left(\mathbf{e}_\delta \cdot \tilde{\mathbf{e}}_\beta\right)$$

Expand, use symmetry of stress tensor $\sigma_{ij} = \sigma_{ji}$:

Transformation of stress components III

$$\tilde{\sigma}_{11} = \sigma_{11} \underbrace{\left(\mathbf{e}_1 \cdot \mathbf{e}_1\right)}_{\cos\alpha} \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1\right) + \sigma_{12} \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1\right) \underbrace{\left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1\right)}_{\sin\alpha} + \sigma_{21} \left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1\right) \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1\right) + \sigma_{22} \left(\mathbf{e}_2 \cdot \mathbf{e}_1\right) \left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1\right.$$

$$= \sigma_{11} \cos^2\alpha + 2\sigma_{12} \sin\alpha \cos\alpha + \sigma_{22} \sin^2\alpha \quad (3)$$

$$\tilde{\sigma}_{12} = \sigma_{11} \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1\right) \underbrace{\left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2\right)}_{-\sin\alpha} + \sigma_{12} \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_1\right) \underbrace{\left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2\right)}_{\cos\alpha} + \sigma_{21} \left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_1\right) \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2\right) + \sigma_{22} \left(\mathbf{e}_2 \cdot \mathbf{e}_1\right) \left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2\right.$$

$$= -\sigma_{11} \sin\alpha \cos\alpha + \sigma_{12} \cos^2\alpha - \sigma_{21} \sin^2\alpha + \sigma_{22} \sin\alpha \cos\alpha$$

$$= \left(\sigma_{22} - \sigma_{11}\right) \sin\alpha \cos\alpha + \left(\cos^2\alpha - \sin^2\alpha\right) \sigma_{12} \quad (4)$$

$$\tilde{\sigma}_{22} = \sigma_{11} \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2\right) \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2\right) + \sigma_{12} \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2\right) \left(\mathbf{e}_2 \cdot \mathbf{e}_2\right) + \sigma_{21} \left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2\right) \left(\mathbf{e}_1 \cdot \tilde{\mathbf{e}}_2\right) + \sigma_{22} \left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2\right) \left(\mathbf{e}_2 \cdot \tilde{\mathbf{e}}_2\right)$$

$$= \sigma_{11} \sin^2\alpha - 2\sigma_{12} \sin\alpha \cos\alpha + \sigma_{22} \cos^2\alpha \quad (5)$$

Transformation of stress components IV

Further simplify using trigonometric relations:

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha, \quad \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \tag{6}$$

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha), \quad \cos^2 \alpha = \tfrac{1}{2}(1 + \cos 2\alpha) \tag{7}$$

Definition

Stress transformation equations in 2D:

$$\tilde{\sigma}_{11} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\alpha + \sigma_{12} \sin 2\alpha \tag{8}$$

$$\tilde{\sigma}_{22} = \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\alpha - \sigma_{12} \sin 2\alpha \tag{9}$$

$$\tilde{\sigma}_{12} = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\alpha + \sigma_{12} \cos 2\alpha \tag{10}$$

Principal stresses and directions I

Given the components of the stress tensor in a given coordinate system, the determination of the maximum normal and shear stresses is critical for the design of structures. The normal and shear stress components on a plane with normal $\mathbf{n}$ are given by:

$$
\begin{aligned}
t_N &= \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} \\
&= \sigma_{ki} n_k n_i \\
t_S &= \sqrt{\|\mathbf{t}^{(\mathbf{n})}\|^2 - t_N^2}
\end{aligned}
$$

It is obvious from these equations that the normal component achieves its maximum $t_N = \|\mathbf{t}^{(\mathbf{n})}\|$ when the shear components are zero. In this case:

$$
\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\sigma} = \lambda \mathbf{n} = \lambda \mathbf{n} \cdot \mathbf{I}
$$

$$
\mathbf{n} \cdot (\boldsymbol{\sigma} - \lambda \mathbf{I}) = \mathbf{0}
$$

where $\mathbf{I} = \delta_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ is the 2nd order identity tensor. In components:

$$n_k(\sigma_{ki} - \lambda\delta_{ki}) = 0 \tag{11}$$

which means that the principal stresses are obtained by solving the previous eigenvalue problem, the principal directions are the eigenvectors of the problem. The eigenvalues λ are obtained by noticing that the last identity can be satisfied for non-trivial $\mathbf{n}$ only if the factor is singular, i.e., if its determinant vanishes:

$$|\sigma_{ij} - \lambda\delta_{ij}| = \begin{matrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{matrix} = 0$$

which leads to the *characteristic equation*:

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$$

where:

$$
\begin{aligned}
I_1 &= \mathrm{tr}[\boldsymbol{\sigma}] \\
&= \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} && (12) \\
I_2 &= \mathrm{tr}[\boldsymbol{\sigma}^{-1}]\mathrm{det}[\boldsymbol{\sigma}] = \frac{1}{2}[\mathrm{tr}[\boldsymbol{\sigma}]^2 - \mathrm{tr}[\boldsymbol{\sigma}^2]] = \\
&= \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right) = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \left(\sigma_{12}\sigma_{21} + \sigma_{23}\sigma_{32} + \sigma_{31}\sigma_{13}\right) && (13) \\
I_3 &= \mathrm{det}[\boldsymbol{\sigma}] = \epsilon_{ijk}\sigma_{i1}\sigma_{j2}\sigma_{k3} \\
&= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{13}^2\sigma_{22} && (14)
\end{aligned}
$$

are called the *stress invariants* because they do not depend on the coordinate system of choice.

Conclusion:

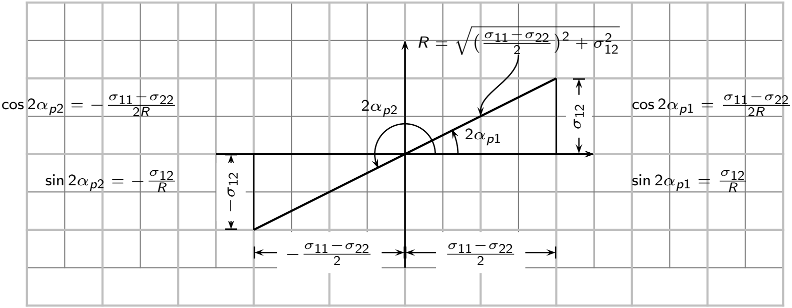
- The principal stresses are the eigenvalues of the matrix of stress tensor components given on any basis.
- The principal directions are the corresponding eigenvectors
- Since σ is symmetric, its eigenvalues are real and its eigenvectors are orthogonal

In the 2D case, we have a less mathematical and more intuitive approach to find principal stresses and directions:

Principal stresses and directions in 2D (plane stress) states: We seek to find direction(s) in which normal stresses are maximized. From (8):

$$\frac{d\sigma_{11}(\alpha)}{d\alpha} = 0 = -\cancel{2}\frac{\sigma_{11} - \sigma_{22}}{\cancel{2}} \sin 2\alpha_p + 2\sigma_{12} \cos 2\alpha_p, \rightarrow \boxed{\tan 2\alpha_p = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}}$$

Interestingly, we find that the condition of maximizing the normal stress coincides with the condition of a zero shear stress according to (10).
This equation has two roots for $2\alpha_p$ which are 180° apart (see figure), which means that α_{p1} and α_{p2} are 90° apart.



In order to obtain the actual value(s) of the maximum stress component σ_{11}, we must replace the values of $\alpha_{p1,2}$ in (8). The sine and cosine of the angles can be obtained from the triangles of the figure where $\frac{\sigma_{11}-\sigma_{22}}{2}, \sigma_{12}$ are assumed both positive or both negative. The diagonals of these triangles have a size $R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$

Principal stresses and directions VII

If we substitute α_{p1} or α_{p2} in (8), we obtain:

$$
\begin{aligned}
\sigma_{\mathrm{I}} = \tilde{\sigma}_{11}(\alpha_{p1}) &= \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\frac{\sigma_{11}-\sigma_{22}}{2R} + \sigma_{12}\frac{\sigma_{12}}{R} \\
&= \frac{\sigma_{11}+\sigma_{22}}{2} + R \\
\sigma_{\mathrm{II}} = \tilde{\sigma}_{22}(\alpha_{p1}) &= \frac{\sigma_{11}+\sigma_{22}}{2} - \frac{\sigma_{11}-\sigma_{22}}{2}\frac{\sigma_{11}-\sigma_{22}}{2R} - \sigma_{12}\frac{\sigma_{12}}{R} \\
&= \frac{\sigma_{11}+\sigma_{22}}{2} - R
\end{aligned}
$$

$$
\tilde{\sigma}_{12} = -\frac{\sigma_{11}-\sigma_{22}}{2}\frac{\sigma_{12}}{R} + \sigma_{12}\frac{\sigma_{11}-\sigma_{22}}{2R} = 0
$$

In summary we have:

Principal stresses and directions in 2D

Given stress components $\sigma_{11}, \sigma_{12}, \sigma_{22}$ in a cartesian basis. The principal stresses and directions are given by:

$$\boxed{\sigma_{\mathrm{I,II}} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}}$$

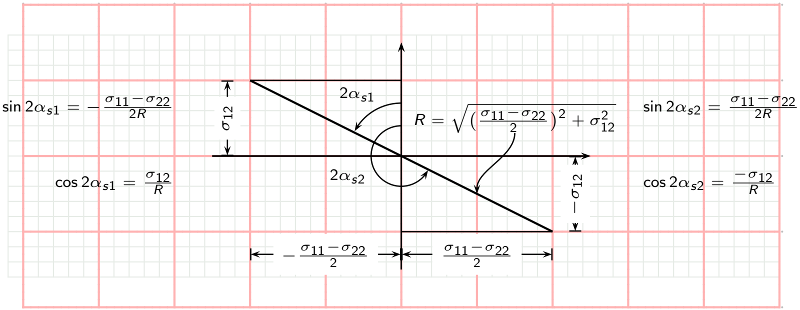
$$\boxed{\tan 2\alpha_p = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}}$$

The maximum shear stress $\sigma_s = \tilde{\sigma}_{12}^{max}$ and the orientation of the planes where they take place α_s can be determined by finding the angle for which the derivative of the shear stress (10) vanishes:

$$\frac{d\tilde{\sigma}_{12}(\alpha)}{d\alpha} = 0 = -\cancel{2}\frac{\sigma_{11} - \sigma_{22}}{\cancel{2}} \cos 2\alpha_s - 2\sigma_{12} \sin 2\alpha_s, \rightarrow \boxed{\tan 2\alpha_s = -\frac{\sigma_{11} - \sigma_{22}}{2\sigma_{12}}}$$

We note that $\tan 2\alpha_s = -\frac{1}{\tan 2\alpha_p}$, i.e., the angle $2\alpha_s$ is at 90° of the angle $2\alpha_p$, that is: **the planes for maximum shear are oriented at $\pm 45^\circ$ from the planes corresponding to the principal directions of stress** Using either one of the two roots of the equation above we can obtain the value of σ_s. A convenient way of doing this is to use directly the values of $\cos 2\alpha_s, \sin 2\alpha_s$ which can be graphically obtained from the figure:

Maximum in-plane shear stress II



Substituting in (8)-(10), we obtain:

$$\tilde{\sigma}_{11}(\alpha_{s1}) = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\frac{\sigma_{12}}{R} + \sigma_{12}\Big(-\frac{\sigma_{11}-\sigma_{22}}{2R}\Big) = \frac{\sigma_{11}+\sigma_{22}}{2}$$

$$\tilde{\sigma}_{22}(\alpha_{s1}) = \frac{\sigma_{11}+\sigma_{22}}{2} - \frac{\sigma_{11}-\sigma_{22}}{2}\frac{\sigma_{12}}{R} - \sigma_{12}\Big(-\frac{\sigma_{11}-\sigma_{22}}{2R}\Big) = \frac{\sigma_{11}+\sigma_{22}}{2}$$

$$\sigma_s = \tilde{\sigma}_{12}(\alpha_{s1}) = -\frac{\sigma_{11}-\sigma_{22}}{2}\Big(-\frac{\sigma_{11}-\sigma_{22}}{2R}\Big) + \sigma_{12}\frac{\sigma_{12}}{R} = \frac{1}{\cancel{R}}\underbrace{\Big[\Big(\frac{\sigma_{11}-\sigma_{22}}{2}\Big)^2 + \sigma_{12}^2\Big]}_{R\cancel{^2}} = R$$

In summary we have:

Maximum shear stresses and directions in 2D

Given stress components $\sigma_{11}, \sigma_{12}, \sigma_{22}$ in a cartesian basis. The maximum shear stresses and directions are given by:

$$\boxed{\sigma_s = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}}$$

$$\boxed{\tan 2\alpha_s = -\frac{\sigma_{11} - \sigma_{22}}{2\sigma_{12}}}$$

On those planes, the normal stresses are the average stress:

$$\tilde{\sigma}_{11}(\alpha_s) = \tilde{\sigma}_{22}(\alpha_s) = \frac{\sigma_{11} + \sigma_{22}}{2}$$

is a graphical representation of the state of stress that is easy to construct, use and remember. The main advantage of Mohr's circle is that it provides significant insight about the physical implications about specific states of stress. The derivation is based on observing that equations (8) and (10) give the parametric representation of a circle in the $\sigma = \sigma_{11}, \tau = \sigma_{12}$ plane centered at $\frac{\sigma_{11}+\sigma_{22}}{2}, 0$ of radius $R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$. This can be seen by eliminating the parameter α from those equations as follows:

$$\tilde{\sigma}_{11} - \frac{\sigma_{11}+\sigma_{22}}{2} = \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\alpha + \sigma_{12}\sin 2\alpha$$

$$\tilde{\sigma}_{12} = -\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\alpha + \sigma_{12}\cos 2\alpha$$

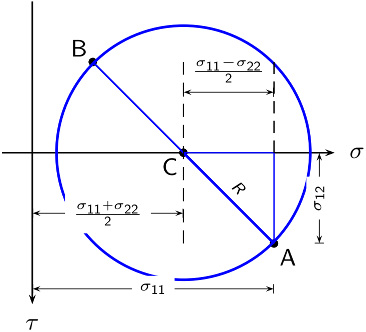
Squaring each equation and adding:

$$\left[\tilde{\sigma}_{11} - \frac{\sigma_{11}+\sigma_{22}}{2}\right]^2 + \tilde{\sigma}_{12}^2 = \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2$$

or

$$[\tilde{\sigma}_{11} - \sigma_{avg}]^2 + \tilde{\sigma}_{12}^2 = R^2$$

giving the equation of the circle mentioned above.

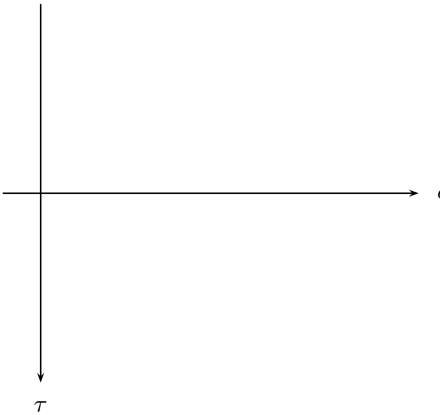


We note that $\alpha = 0$ corresponds to point A in the figure and that as α increases from 0 to $\pi/2$, the various points of the circle correspond to the values of $\tilde{\sigma}_{11}(\alpha), \tilde{\sigma}_{12}(\alpha)$.
For $\alpha = \pi/2$ we reach point B. When we rotate our axes by this angle, we can see that the stress components are $\tilde{\sigma}_{11} = \sigma_{22}, \tilde{\sigma}_{12} = -\sigma_{12}$ (and $\tilde{\sigma}_{22} = \sigma_{11}$). We can also see that a rotation of value α of the cartesian plane normal (basis vectors) corresponds to a rotation 2α in the circle in the same direction.

Mohr's circle construction procedure and uses

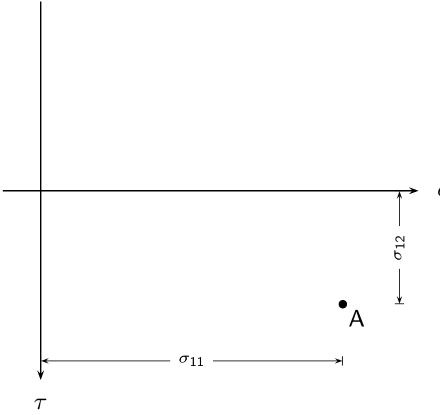
Construction of Mohr's circle

- Set up axes: σ abscissa, τ ordinate (positive pointing down)



Construction of Mohr's circle

- Set up axes: σ abscissa, τ ordinate (positive pointing down)
- Plot reference point A$= (\sigma_{11}, \sigma_{12})$ which represents the stress components on a plane with a normal $\mathbf{e}_1$ $(\alpha = 0)$



Mohr's circle construction procedure and uses

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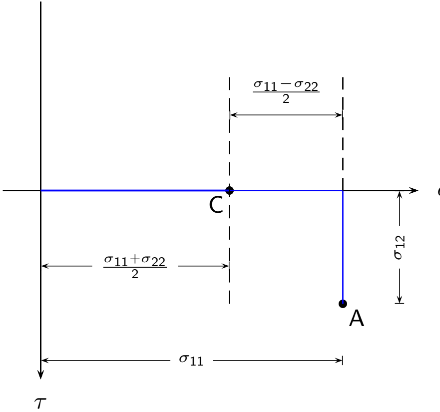
- Set up axes: σ abscissa, τ ordinate (positive pointing down)
- Plot reference point A$= (\sigma_{11}, \sigma_{12})$ which represents the stress components on a plane with a normal $\mathbf{e}_1$ $(\alpha = 0)$
- Plot circle's center C at point $\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right)$



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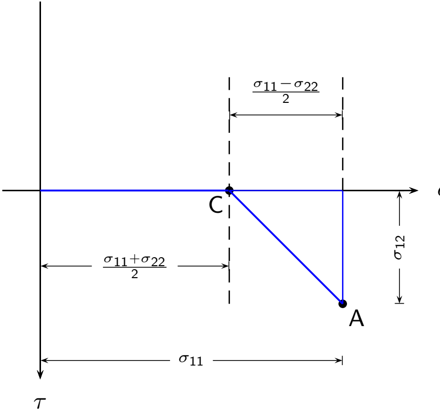
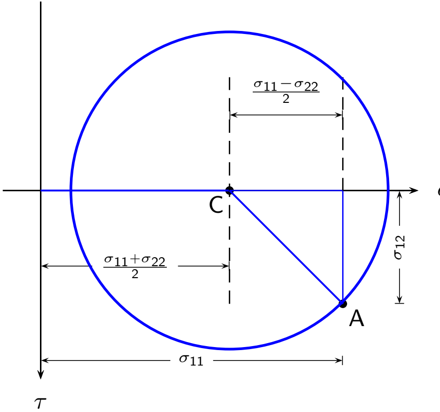
- The distance between points C and A is the radius of the circle R



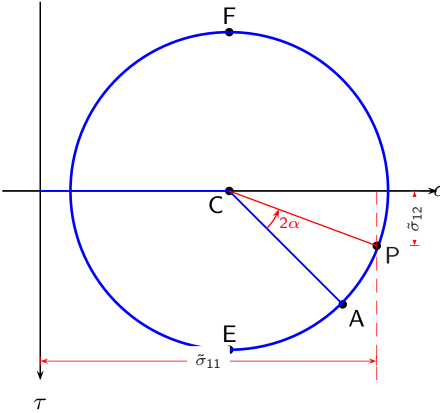
Construction of Mohr's circle

- Set up axes: σ abscissa, τ ordinate (positive pointing down)
- Plot reference point A$= (\sigma_{11}, \sigma_{12})$ which represents the stress components on a plane with a normal $\mathbf{e}_1$ $(\alpha = 0)$
- Plot circle's center C at point $\left(\frac{\sigma_{11}+\sigma_{22}}{2}, 0\right)$
- The distance between points C and A is the radius of the circle R
- sketch the circle centered at C with radius R.



Stresses on arbitrary plane α

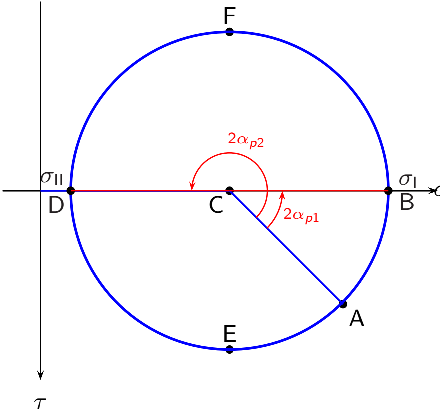
- draw a line from C at an angle α from line CA. The coordinates of the point P where this line intersects the circle gives $\tilde{\sigma}_{11}, \tilde{\sigma}_{12}$
- these coordinates can be obtained from the figure using trigonometry.



Mohr's circle: construction procedure and uses

Principal Stresses and directions

- σ_{I} and σ_{II} ($\sigma_{\text{I}} \geq \sigma_{\text{II}}$) correspond to points B and D in the circle where there are no shear stresses.
- the angles of the principal planes appear on the circle as $2\alpha_I, 2\alpha_{\text{II}}$ measured from the line CA to lines CB and CD, respectively



Maximum in-plane shear stress and directions

- Obviously, points E and F determine the maximum shear stress $\sigma_s = \pm R$.
- The angles $2\alpha_{s1}, 2\alpha_{s2}$ are measured from line CA to lines CE and CF, respectively. For point E, the angle of rotation is clockwise, and so must be the rotation of the element.
- It is clear that in both orientations the normal stress is the average stress.