Problem 1. Consider the Cournot competition game from the last problem set. Now suppose this game is infinitely repeated with common discount factor $\delta < 1$.

(a) What is the set of feasible and strictly individually rational payoffs in this game? In particular, what is the greatest symmetric feasible payoff vector?

Solution. Based on Alex's clarification and hint, we only needed you to give the highest symmetric payoff vector. This is $(u_1, u_2) = (1/8, 1/8)$. The players can get this payoff vector by choosing $q_1 = q_2 = 1/4$.

For completeness, we include the solution to the rest of the question as asked. We first argue that the minmax payoff for each player is equal to zero. Player i can guarantee herself a payoff of zero by choosing $q_i = 0$. She cannot guarantee herself any higher payoff: if player j chooses $q_j = 1$, then the best response of player i is to choose $q_i = 0$, which leads to a payoff of zero.

It is also easy to see that $(u_1, u_2) = (0, 1/4)$ and $(u_1, u_2) = (1/4, 0)$ are feasible payoffs: if player i chooses $q_i = 0$, then the best response of player j is to choose $q_j = 1/2$, which leads to payoffs $u_i = 0$ and $u_j = 1/4$. Therefore, the set $S = \{(u_1, u_2) : u_1, u_2 > 0, u_1 + u_2 \leq 1/4\}$ is a subset of the set of feasible and strictly individually rational payoffs.

We next argue that any $(u_1, u_2) \notin S$ is either not feasible or not strictly individually rational. Since the minmax payoff is zero for both players, any (u_1, u_2) such that $u_i \leq 0$ for some i is not strictly individually rational. Suppose next that $u_1 + u_2 > 1/4$. This implies that $(q_1 + q_2)(1 - q_1 - q_2) > 1/4$, which is a contradiction. Therefore, no (u_1, u_2) such that $u_1 + u_2 > 1/4$ is feasible.

(b) Show that, using trigger strategies which involve switching to the static Nash equilibrium $\left(q_1 = \frac{1}{3}, q_2 = \frac{1}{3}\right)$, the players can attain their greatest symmetric feasible payoffs in a SPE whenever $\delta \geq \frac{9}{17}$.

Solution. Consider a trigger strategy of the following form. Players choose $q_1 = q_2 = 1/4$ as long as both players have chosen $q_1 = q_2 = 1/4$ in all the previous periods; they switch to the static Nash equilibrium if any player deviates. We show that this is a SPE whenever $\delta \geq \frac{9}{17}$. Since the players play the static Nash equilibrium following a deviation, we only need to check that they do not have profitable deviations along the path of play.

The payoff to player i from choosing $q_i = 1/4$ is given by

$$\frac{1}{8} + \delta\frac{1}{8} + \delta^2\frac{1}{8} + \cdots = \frac{1}{1-\delta}\frac{1}{8}.$$

If she deviates to choosing q_i' instead, then her payoff is given by

$$q_i'\left(1 - q_i' - \frac{1}{4}\right) + \delta\frac{1}{9} + \delta^2\frac{1}{9} + \cdots = q_i'\left(1 - q_i' - \frac{1}{4}\right) + \frac{\delta}{1-\delta}\frac{1}{9}.$$

The most profitable deviation available to player i is to choose the (static) best response to $q_j = 1/4$, i.e., to choose $q_i' = 3/8$. This leads to a payoff of

$$\frac{9}{64} + \frac{\delta}{1-\delta}\frac{1}{9}.$$

Hence player i does not have a profitable deviation as long as

$$\frac{1}{1-\delta}\frac{1}{8} \geq \frac{9}{64} + \frac{\delta}{1-\delta}\frac{1}{9},$$

which is equivalent to

$$\delta \geq \frac{9}{17}.$$

(c) We will now show that the set of feasible and individually rational payoff vectors for the Cournot game includes some where the players receive *lower* payoffs than they would if they repeatedly played the static Nash equilibrium. Use the one-shot deviation principle to show that the symmetric strategy profile described below is a SPE if $\delta \geq \frac{9}{40}$, and show that the per-period payoffs players receive from it corresponds to a payoff vector in which each player receives less than the static Nash equilibrium profit.

$$q_i(h^t) = \begin{cases} 3/8 & \text{if } t = 0 \\ 1/3 & \text{if } t > 0 \text{ and both players followed strategy } q \text{ in the previous period} \\ 3/8 & \text{otherwise} \end{cases}$$

(Note that the "punishment" of $q_1 = q_2 = \frac{3}{8}$ lasts for only one period after a deviation.)

Solution. Since $q_1 = q_2 = 1/3$ is a static Nash equilibrium, we only need to check the players' deviations in histories in which they are supposed to play $q_1 = q_2 = 3/8$. If players follow the strategy, their payoff is given by

$$\frac{3}{32} + \delta\frac{1}{9} + \delta^2\frac{1}{9} + \cdots = \frac{3}{32} + \frac{\delta}{1-\delta}\frac{1}{9},$$

whereas if player i deviates to q_i', her payoff is given by

$$q_i'\left(1 - q_i' - \frac{3}{8}\right) + \delta\frac{3}{32} + \delta^2\frac{1}{9} + \delta^3\frac{1}{9} + \cdots = q_i'\left(1 - q_i' - \frac{3}{8}\right) + \delta\frac{3}{32} + \frac{\delta^2}{1-\delta}\frac{1}{9}.$$

The most profitable deviation for player i is to choose $q_i' = 5/16$, leading to a payoff of

$$\frac{25}{256} + \delta\frac{3}{32} + \delta^2\frac{1}{9} + \delta^3\frac{1}{9} + \cdots = \frac{25}{256} + \delta\frac{3}{32} + \frac{\delta^2}{1-\delta}\frac{1}{9}.$$

Therefore, player i will not have a profitable deviation if

$$\frac{3}{32} + \frac{\delta}{1-\delta}\frac{1}{9} \geq \frac{25}{256} + \delta\frac{3}{32} + \frac{\delta^2}{1-\delta}\frac{1}{9},$$

which is equivalent to $\delta \geq \frac{9}{40}$.

The per-period payoff each player receives from the strategy profile is given by

$$(1-\delta)\left(\frac{3}{32} + \frac{\delta}{1-\delta}\frac{1}{9}\right) = (1-\delta)\frac{3}{32} + \delta\frac{1}{9},$$

which is smaller that 1/9, the static Nash equilibrium profit.

(d) Construct a SPE using the strategies in part (c) as punishments where the players attain their greatest symmetric feasible payoffs in a SPE for some $\delta < \frac{9}{17}$.

Solution. We consider the following strategy profile. Both players choose $q_1 = q_2 = 1/4$ as long as both players have chosen $q_1 = q_2 = 1/4$ in all the previous periods. If there is a deviation, players switch to playing the strategy profile described in part (c).

By the argument in part (c), as long as $\delta \geq \frac{9}{40}$, players have no profitable deviation in subgames in which some players have deviated from $q_1 = q_2 = 1/4$. So to prove that the strategy profile is a SPE, we only need to argue that players have no profitable deviation along the path of play. As we saw in part (b), the payoff from choosing $q_i = 1/4$ is given by $\frac{1}{1-\delta}\frac{1}{8}$. By the computation in part (c), the payoff from deviating to q_i' is given by

$$q_i'\left(1 - q_i' - \frac{1}{4}\right) + \delta\frac{3}{32} + \frac{\delta^2}{1-\delta}\frac{1}{9}.$$

The most profitable deviation for player i is to choose $q_i' = 3/8$, leading to a payoff of

$$\frac{9}{64} + \delta\frac{3}{32} + \frac{\delta^2}{1-\delta}\frac{1}{9}.$$

Therefore, the strategy profile is a SPE as long as $\delta \geq \frac{9}{40}$ and

$$\frac{1}{1-\delta}\frac{1}{8} \geq \frac{9}{64} + \delta\frac{3}{32} + \frac{\delta^2}{1-\delta}\frac{1}{9},$$

which is equivalent to

$$\delta \geq \frac{27}{20} - \frac{3\sqrt{41}}{20} > \frac{9}{40}.$$

The strategy profile is a SPE for $\delta \geq \frac{27}{20} - \frac{3\sqrt{41}}{20} < \frac{9}{17}$ and the players play $q_1 = q_2 = 1/4$ along the path of play and get their greatest symmetric feasible payoffs, as desired.

Problem 2. In Lectures 19-20 we saw that, in the repeated prisoners' dilemma with anonymous random matching, for any fixed N, there exists $\bar{\delta}$ such that, if $\delta > \bar{\delta}$, *Always Cooperate* is a Nash equilibrium outcome. Prove that, for any fixed δ, there exists $\bar{N}$ such that, if $N > \bar{N}$, the unique Nash equilibrium outcome is *Always Defect.* So, with anonymous random matching, is cooperation possible in a large group of patient players, or isn't it?

Solution. Fix an arbitrary Nash equilibrium, an arbitrary $\epsilon > 0$, and an arbitrary history h^T, and consider a deviation for player i in period T at history h^T. The number of players who may know of player i's deviation t periods after the deviation is upper bounded by 2^{t-1}. So the probability that in period $t+T$ player i is matched with a player who is aware of player i's deviation is bounded above by $\max\{1, 2^{t-1}/N\}$. Player i's per-period payoff loss from deviating is bounded above by 3. Therefore, the loss in the present-discounted payoff of player i from deviating is bounded above by

$$\delta \sum_{t=1}^{\infty} \max\left\{1, \frac{2^{t-1}}{N}\right\} \delta^{t-1} 3.$$

For any $\delta < 1$, there exists some $\bar{N}$ such that the above expression is smaller than ϵ for all $N > \bar{N}$. That is, for any $\delta < 1$, the loss in future utility to player i from deviating is smaller than $\epsilon > 0$ if $N > \bar{N}$. Therefore, when $N > \bar{N}$, if player i plays cooperate at history h^T with probability p, then she can gain at least $p - \epsilon$ by deviating to defect. Since $\epsilon > 0$ is arbitrary, the player must play cooperate with probability zero at history h^T. Since h^T is arbitrary, the player must follow the strategy *Always Defect.* Whether cooperation is possible in a large group of patient players therefore depends on how patient players are relative to the size of the society.