Problem 1. Suppose we have a set of 3 sellers labeled a, b, and c, and a set of 3 buyers labeled x, y, and z. Each seller is offering a distinct house for sale, and the valuations of the buyers for the houses are as follows.

Buyer	Value for a's house	Value for b's house	Value for c's house
x	7	7	7
y	7	6	3
z	5	4	3

(a) Suppose that a charges a price of 4 for his house, b charges a price of 3 for his house, and c charges a price of 1. Is this set of prices market-clearing? Give an explanation for your answer, using the relevant definitions.

(b) Describe what happens if we run the bipartite graph auction procedure to determine market-clearing prices, by saying what the prices are at the end of each round of the auction, including what the final market-clearing prices are when the auction comes to an end. *[Note: In some rounds, you may notice that there are multiple choices for the constricted set of buyers. Under the rules of the auction, you can choose any such constricted set.]*

Solution.

(a) No for the prices to be market-clearing each buyer must prefer the good they receive at the price. Both buyers x and z strictly prefer house c at this price vector so this is not market-clearing.

(b) Prices start at $(a, b, c) = (0, 0, 0)$. x has payoff $(7, 7, 7)$, y has payoff $(7, 6, 3)$ and z has payoff $(5, 4, 3)$. Buyer set $\{y, z\}$ is constricted as both prefer a. So we increase the price of a. Now prices are $(a, b, c) = (1, 0, 0)$. x has payoff $(6, 7, 7)$, y has payoff $(6, 6, 3)$ and z has payoff $(4, 4, 3)$. No buyer set is constricted and we assign c to x, b to y and a to z.

Problem 2. An important model of competition between firms in economics is *Cournot competition*, where firms compete by choosing how much output to produce, and the resulting price is determined by the market. In the simplest example of Cournot competition, each of 2 firms, $i = 1, 2$, chooses quantity $q_i \in [0, 1]$, and the resulting market price is $1 - q_1 - q_2$. Thus, if firm 1 produces q_1 and firm 2 produces q_2, their payoffs are $q_1 (1 - q_1 - q_2)$ and $q_2 (1 - q_1 - q_2)$, respectively.

(a) Find a PSNE in this game. Prove that it is the only one.

A closely related model is *Stackelberg competition*. Under Stackelberg competition, the firms' payoffs as a function of q_1 and q_2 are the same as above. The difference is that now the quantities are set sequentially rather than simultaneously: first firm 1 chooses q_1, and then—after observing q_1—firm 2 chooses q_2.

(b) Solve for an equilibrium in which the second firm optimizes conditional on what the first does, and the first firm takes this into account when moving first. This is a pure-strategy subgame perfect equilibrium (SPE) in this game. Prove that it is the only one.

(c) Explain intuitively why the predicted outcome is different in (a) and (b). Why does firm 1 have a "first-mover advantage" in Stackelberg competition?

Solution.

(a) Since the payoff of firm i is strictly concave in q_i, the first-order optimality condition fully characterizes firm i's best response:

$$\frac{\partial}{\partial q_i} \left(q_i \left(1 - q_i - q_j\right)\right) = 1 - 2q_i - q_j = 0.$$

Firm i's best-response function is thus given by

$$q_i^*(q_j) = \frac{1 - q_j}{2}.$$

The unique solution to the above equations for $i = 1, 2$ is given by $q_1^* = q_2^* = \frac{1}{3}$.

(b) We can find the SPE using backward induction. Given firm 1's choice q_1, firm 2's best-response function is given by $q_2^*(q_1) = \frac{1-q_1}{2}$. The payoff to firm 1 from choosing q_1—taking into account that firm 2 will best respond—is given by

$$q_1 \left(1 - q_1 - \frac{1 - q_1}{2}\right) = \frac{1}{2} q_1 (1 - q_1).$$

This is a strictly concave function, so firm 1's optimal choice is characterized by the following first-order condition:

$$\frac{\partial}{\partial q_1}\left(\frac{1}{2}q_1(1-q_1)\right)=\frac{1}{2}-q_1=0.$$

Therefore, $q_1^*=\frac{1}{2}$. In the unique SPE, firm 1 chooses $q_1^*=\frac{1}{2}$, and firm 2 sets $q_2^*(q_1)=\frac{1-q_1}{2}$ in the subgame in which firm 1 has chosen q_1. [Note that firm 2's strategy is a full contingency plan that describes what the firm does following any choice of q_1 by firm 1.]

(c) Intuitively, Firm 1 gets a first-mover advantage because moving first gives it the ability to make a credible commitment to produce $q_1=\frac{1}{2}$ against $q_2=\frac{1}{4}$. In the simultaneous game, this is not credible since it would be optimal to deviate to $q_1=\frac{3}{8}$.

Problem 3. Consider a variant of the alternating-offers bargaining model discussed in Lecture 17 where, instead of the seller and buyer taking turns making offers, in each period one of the two parties is randomly selected to make the offer in that period. (That is, if the parties do not reach an agreement in period t, in period $t+1$ a fair coin is flipped to determine who makes the offer in period $t+1$.)

(a) Show that, if the buyer offers $p_B = \delta_S \left(\frac{1}{2}p_S + \frac{1}{2}p_B\right)$, that the seller is indifferent between selling and not selling. Argue that, if the buyer offers this price and the seller accepts it, as neither party has a strictly profitable deviation.

(b) Show that, if the seller offers p_S, such that $1 - p_S = \delta_B \left(1 - \frac{1}{2}p_S - \frac{1}{2}p_B\right)$, that the buyer is indifferent. Argue that the buyer accepts it, as neither party has profitable deviation.

(c) Compare this answer to the one we derived in lectures for alternating offers bargaining, where S moves first. Is the advantage from being the first mover larger or smaller? Intuitively, why?

Solution. (a) If the seller accepts, they get payoff $\delta_S \left(\frac{1}{2}p_S + \frac{1}{2}p_B\right)$. If the seller declines, then in the next stage with probability $\frac{1}{2}$ it is the buyers turn and they offer p_B and with probability $\frac{1}{2}$ it is the seller's turn and they offer p_S. Now the sellers discounted expected payoff is $\delta_S \left(\frac{1}{2}p_S + \frac{1}{2}p_B\right)$ which is the same as before. So there is no profitable deviation for the seller.

(b) If the seller offers a price p_S such that $1 - p_S = \delta_B \left(1 - \frac{1}{2}p_S - \frac{1}{2}p_B\right)$. If the buyer accepts then their payoff is $(1 - p_S)$. If the buyer rejects we move to the next round where with probability $\frac{1}{2}$ it is the seller's turn and they again offer p_S or it is the buyer's turn with probability $\frac{1}{2}$ and they offer p_B. Thus the expected payoff is $\delta_B \left(1 - \frac{1}{2}p_B - \frac{1}{2}p_S\right)$, which is the same payoff as before. Thus, the buyer will accept the price p_S as there is no profitable deviation for the buyer.

(c) In the unique SPE, the seller offers price p_S whenever he is the one making an offer and accepts any price larger than or equal to p_B, and the buyer offers price p_B whenever she is the one making an offer and accepts any price smaller than or equal to p_S, where

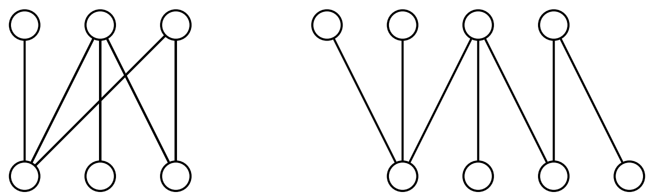
$$\begin{aligned} p_B &= \delta_S \left(\frac{1}{2}p_S + \frac{1}{2}p_B\right), \\ 1 - p_S &= \delta_B \left(1 - \frac{1}{2}p_S - \frac{1}{2}p_B\right). \end{aligned}$$

Solving the above system of equations for p_B and p_S, we get

$$p_B = \frac{\delta_S(1-\delta_B)}{2-\delta_B-\delta_S},$$
$$p_S = \frac{(2-\delta_S)(1-\delta_B)}{2-\delta_B-\delta_S} > p_B.$$

The advantage from being the one to make the offer is larger than it is in alternating-offers bargaining. Intuitively, this is because now the player who makes an offer in the current period may have the chance to make an offer in the next period.

Problem 4. Find the predicted payoffs in the networks below using the Corominas-Bosch model. Buyers are the upper nodes and sellers are the lower nodes. [Note: The "Corominas-Bosch model" is exactly the model of bargaining in networks we studied in Lecture 18. Compute the payoffs for the limit where $\delta \to 1$: that is, all over-demanded nodes get payoff 1, all under-demanded nodes get payoff 0, and all perfectly matched nodes get payoff 0.5.]



Solution.