Problem 1 (Commuting and Congestion Taxes). A continuum of commuters of mass 1 must commute from Cambridge to Boston. Half of the commuters have flexible schedules, and their payoff is the negative of the total travel time. Half of the commuters have a tight deadline, and their payoff is the negative of the "worst-case" travel time.

There are two modes of transportation: train and car. The car always takes x hours if fraction x of commuters are on the road. The train takes 1 hour with probability $1 - \epsilon$, and 2 hours with probability ϵ (a rare delay). All commuters make choices before knowing whether there will be a delay.

(a) Find the unique Nash equilibrium. What is the average commuting time? What is the average payoff?

(b) Assume that the government could mandate that each commuter takes a specific route, which is conditioned on whether they have a deadline. What allocation maximizes average payoff?

(c) Could the government implement the allocation from (b) by asking everyone if they have a deadline, and then (randomly, if necessary) assigning each type of person to the road or train?

[Hint: people might lie]

(d) Assume that the government's feasible policy is, instead, to impose a cap on the number of drivers on the road. If commuters want to take the road in excess of the cap, both the flexible and deadline types are equally likely to be turned away and forced to take the train. Can the government implement their preferred solution from (b) with a cap? If so, what cap(s) achieve this?

(e) Assume instead that the government can impose a toll on the roads from Cambridge to Boston. All commuters value their time at $10 per hour. Can the government implement their preferred solution from (b) with a toll? If so, what toll(s) achieve this?

Solution. (a) In the unique Nash equilibrium, everyone takes the road. Each individual has payoff -1. No one has an incentive to deviate — flexible people would get payoff $-((1-\epsilon)(1)+(\epsilon)(2)) = -(1+\epsilon)$ on the train and tight-deadline people would get payoff -2. The average commuting time and average payoff in the equiligrium are both -1.

(b) The government would put all the tight-deadline individuals on the road; $\epsilon/2$ mass of the flexible people on the road; and the remaining $1/2 - \epsilon/2$ people on the train. If ϵ is very small, of course, this is approximately the same as putting all the flexible people on the train.

(c) No — all the flexible commuters will say they have a deadline so they can be put on the road.

(d) The only possibility is that the government caps the road at $1/2 + \epsilon/2$ drivers. But everyone would show up to the road, and people would have to be randomly turned away. There is no way to make sure that only tight-deadline commuters end up on the road.

(e) Here, we would accept two answers. You could say "no" because the government will never get $\epsilon/2$ of the flexible commuters onto the road. But they can implement something very close, if ϵ is small. The government can charge a tax that makes driving prohibitively costly for flexible commuters but still attractive for tight-deadline commuters. In particular, to dissuade flexible drivers, the tax τ in units of "hours" must obey $1/2 + \tau > 1 + \epsilon$ or $\tau > 1/2 + \epsilon$. To keep tight-deadline commuters, the tax must obey $1/2 + \tau < 2$ or $\tau < 3/2$. Thus any tax $1/2 + \epsilon < \tau < 3/2$ would put all tight-deadline commuters on the road and all flexible commuters on the train. In dollars, this is $5 + 10\epsilon < \tau < 15$.

Problem 2 (Easley and Kleinberg Chapter 17, Exercise 1)**.** Consider a product that has network effects in the sense of our model from Chapter 17. Consumers are named using real numbers between 0 and 1; the reservation price for consumer x when a fraction z of the population uses the product is given by the formula $r(x)f(z)$, where $r(x) = 1 - x$ and $f(z) = z$.

(a) Let's suppose that this good is sold at cost $\frac{1}{4}$ to any consumer who wants to buy a unit. What are the possible equilibrium numbers of purchasers of the good?

Solution. The payoff to the consumer labeled x from buying the good at cost c when fraction z of the population uses the product is given by $(1-x)z - c$. So consumer x buys the product if and only if $x \leq 1 - \frac{c}{z}$. The fraction of the population with $x \leq 1 - \frac{c}{z}$ is given by $\max\{1 - \frac{c}{z}, 0\}$. Therefore, any equilibrium must satisfy $z = \max\{1 - \frac{c}{z}, 0\}$. $z = 0$ is always an equilibrium, and so are the roots of the equation $z^2 - z + c = 0$ that belong to the interval $[0, 1]$. When $c = \frac{1}{4}$, there are two equilibria, one with $z = 0$ and the other with $z = \frac{1}{2}$.

(b) Suppose that the cost falls to $\frac{2}{9}$ and that the good is sold at this cost to any consumer who wants to buy a unit. What are the possible equilibrium numbers of purchasers of the good?

Solution. When $c = \frac{2}{9}$, the solutions of equation $z^2 - z + c = 0$ are given by $z = \frac{1}{3}$ and $z = \frac{2}{3}$, so there are three equilibria with $z = 0$, $z = \frac{1}{3}$, and $z = \frac{2}{3}$.

(c) Briefly explain why the answers to parts (a) and (b) are qualitatively different.

Solution. In part (a), the equilibrium with $z > 0$ is at the point of tangency of the curve $\max\{1 - \frac{c}{z}, 0\}$ to the 45-degree line, whereas in part (b), the equilibria with $z > 0$ are at the intersection of the curve described by $\max\{1 - \frac{c}{z}, 0\}$ and the 45-degree line.

(d) Which of the equilibria you found in parts (a) and (b) are stable? Explain your answer.

Solution. $z = 0$ is a stable equilibrium in both parts. The $z = \frac{1}{2}$ equilibrium in part (a) is neither stable nor unstable: starting from $z = \frac{1}{2} + \epsilon$ for $\epsilon > 0$, best-response dynamics will bring z back to $\frac{1}{2}$, whereas starting from $z = \frac{1}{2} + \epsilon$ for $\epsilon < 0$, best-response dynamics will lead z to converge to 0. In part (b), the $z = \frac{2}{3}$ equilibrium is stable and the $z = \frac{1}{3}$ is unstable. This is a consequence of the fact that, when $c = \frac{2}{9}$, the curve $\max\{1 - \frac{c}{z}, 0\}$ crosses the 45-degree line from below at $z = \frac{1}{3}$ and from above at $z = \frac{2}{3}$.

Problem 3 (Network Game)**.** Fix an adjacency matrix g with entries (g_{ij}). Consider the linear best-response game with best responses

$$x_i^* = \begin{cases} 1 - \delta \sum_{j \neq i} g_{ij} x_j & \text{if } \delta \sum_{j \neq i} g_{ij} x_j \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $\delta \in [0, 1)$. Consider the case where g corresponds to the 4-player star network (i.e., one center node and three periphery nodes).

(a) Assume $\delta < \frac{1}{3}$. Find a pure-strategy Nash equilibrim (PSNE) where all players take positive actions ($x_i > 0$ for all i).

(b) Assume $\frac{1}{3} \leq \delta < 1$. Find a PSNE.

(c) Assume $\delta = 1$. Find two different PSNE.

(d) Now consider the 4-player line network: i.e. the network is

$$(1) - (2) - (3) - (4)$$

Assume $\delta = \frac{1}{2}$. Find an PSNE where player 1 and player 4 take one action $x > 0$ and player 2 and player 3 take another action $x' > 0$.

[Hint: such a strategy profile is an equilibrium if and only if $x + \delta x' = 1$ and $x' + \delta x' + \delta x = 1$.]

Solution. (a) Let 1 denote the center node. In the PSNE where all players take positive actions, we must have $x_1^* = 1 - \delta(x_2^* + x_3^* + x_4^*)$ and $x_2^* = x_3^* = x_4^* = 1 - \delta x_1^*$. Solving for x_i^*, we get

$$x_1^* = \frac{1 - 3\delta}{1 - 3\delta^2},$$
$$x_2^* = x_3^* = x_4^* = \frac{1 - \delta}{1 - 3\delta^2}.$$

Note that this a PSNE only if $\delta > \frac{1}{3}$.

(b) When $\frac{1}{3} \leq \delta < 1$, there is no PSNE in which all players take positive actions. Therefore, at least one of the players must choose $x_i = 0$ in equilibrium. Let's start by assuming that the center node chooses $x_1^* = 0$. Then the best responses of periphery nodes are given by $x_2^* = x_3^* = x_4^* = 1$. We only need to check that $x_1^* = 0$ is indeed a best response for player 1. But this is trivially the case since $1 - \delta(x_2^* + x_3^* + x_4^*) = 1 - 3\delta \leq 1$.

(c) When $\delta = 1$, the equilibrium we found in part (b) is still an equilibrium. But now there is also an equilibrium in which $x_1^* = 1$ and $x_2^* = x_3^* = x_4^* = 0$. To verify this note that $x_1^* = 1 - \delta(x_2^* + x_3^* + x_4^*) = 1$, so player 1 is best responding, and $\delta x_1^* = 1 \leq 1$, so players $i \neq 1$ are also best responding.

(d) Using the hint, we only need to find x and x' that solve $x + \delta x' = 1$ and $x' + \delta x' + \delta x = 1$. When $\delta = \frac{1}{2}$, the solution to the above equations is given by $x = \frac{4}{5}$ and $x' = \frac{2}{5}$.