Problem 1 (Commuting and Congestion Taxes)**.** A continuum of commuters of mass 1 must commute from Cambridge to Boston. Half of the commuters have flexible schedules, and their payoff is the negative of the total travel time. Half of the commuters have a tight deadline, and their payoff is the negative of the "worst-case" travel time.

There are two modes of transportation: train and car. The car always takes x hours if fraction x of commuters are on the road. The train takes 1 hour with probability $1-\epsilon$, and 2 hours with probability ϵ (a rare delay). All commuters make choices before knowing whether there will be a delay.

(a) Find the unique Nash equilibrium. What is the average commuting time? What is the average payoff?

(b) Assume that the government could mandate that each commuter takes a specific route, which is conditioned on whether they have a deadline. What allocation maximizes average payoff?

(c) Could the government implement the allocation from (b) by asking everyone if they have a deadline, and then (randomly, if necessary) assigning each type of person to the road or train?

[Hint: people might lie]

(d) Assume that the government's feasible policy is, instead, to impose a cap on the number of drivers on the road. If commuters want to take the road in excess of the cap, both the flexible and deadline types are equally likely to be turned away and forced to take the train. Can the government implement their preferred solution from (b) with a cap? If so, what cap(s) achieve this?

(e) Assume instead that the government can impose a toll on the roads from Cambridge to Boston. All commuters value their time at \$10 per hour. Can the government implement their preferred solution from (b) with a toll? If so, what toll(s) achieve this?

Problem 2 (Easley and Kleinberg Chapter 17, Exercise 1)**.** Consider a product that has network effects in the sense of our model from Chapter 17. Consumers are named using real numbers between 0 and 1; the reservation price for consumer x when a fraction z of the population uses the product is given by the formula $r(x)f(z)$, where $r(x) = 1 - x$ and $f(z) = z$.

(a) Let's suppose that this good is sold at cost $\frac{1}{4}$ to any consumer who wants to buy a unit. What are the possible equilibrium numbers of purchasers of the good?

(b) Suppose that the cost falls to $\frac{2}{9}$ and that the good is sold at this cost to any consumer who wants to buy a unit. What are the possible equilibrium numbers of purchasers of the good?

(c) Briefly explain why the answers to parts (a) and (b) are qualitatively different.

(d) Which of the equilibria you found in parts (a) and (b) are stable? Explain your answer.

Problem 3 (Network Game)**.** Fix an adjacency matrix g with entries (g_{ij}). Consider the linear best-response game with best responses

$$x_i^* = \begin{cases} 1 - \delta \sum_{j \neq i} g_{ij} x_j & \text{if } \delta \sum_{j \neq i} g_{ij} x_j \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

where $\delta \in [0, 1)$. Consider the case where g corresponds to the 4-player star network (i.e., one center node and three periphery nodes).

(a) Assume $\delta < \frac{1}{3}$. Find a pure-strategy Nash equilibrim (PSNE) where all players take positive actions ($x_i > 0$ for all i).

(b) Assume $\frac{1}{3} \leq \delta < 1$. Find a PSNE.

(c) Assume $\delta = 1$. Find two different PSNE.

(d) Now consider the 4-player line network: i.e. the network is

$$(1) - (2) - (3) - (4)$$

Assume $\delta = \frac{1}{2}$. Find an PSNE where player 1 and player 4 take one action $x > 0$ and player 2 and player 3 take another action $x' > 0$.

[Hint: such a strategy profile is an equilibrium if and only if $x + \delta x' = 1$ and $x' + \delta x' + \delta x = 1$.]

Problem 4 (Extra Credit: Schelling's Segregation Model)**.** For up to seven points of extra credit, please answer **one of the following two** problems:

1. *Simulation exercise.* Using a computer language of your choice, code up Schelling's model of segregation on a 50x50 grid. Leave 250, or 10%, of the squares blank. The code will take as inputs the fraction of like-type neighbors that leaves one satisfied, p, and the initial fraction of the first type of agent, q.

   (a) Start with an initial fraction $q = 1/2$ of each type of agent. Plot, for a grid of values of $p \in [0, 1]$, the fraction of one's neighbors who are of the same type in the long-run equilibrium. Comment on the relationship.

   (b) Re-do the exercise with $q = 3/4$. How do your results change?

2. *Segregation in practice.* Read "Tipping and the Dynamics of Segregation," by Card, Mas, and Rothstein (2008). A pdf is posted on Stellar. Answer the following questions about Sections I to IV and VII. You may want to read the questions before reading the paper, so you know what parts to focus on—we obviously don't expect you to absorb every single detail.

   (a) What is the authors' main result, in your reading? Describe this result in three or four sentences.

   (b) Read the authors' description of a theoretical model in Section II. Compare and contrast this model of "tipping points" with Schelling's original model, as we learned it in lecture. Which do you find more descriptive?

   (c) Describe how the authors define "average tolerance" in Section VII. How good of a match is this measure to the notion of preference for homophily in the authors' model (or Schelling's model from lecture, if you prefer that model)? What do you think would be a better measure, if you could do your own survey data collection?