Problem 1 (Morris contagion model). (a) (Exercise 9.16 in Jackson) Consider a network (N, g) where each node take action $a \in \{0, 1\}$, and action 1 is the optimal action for a node if and only if a fraction of at least q of his or her neighbors take action 1. Show that a sufficient condition for never having a contagion from any group of m nodes is to have at least $m+1$ disjoint sets of nodes that are each more than $(1-q)$ cohesive.

(b) Consider a variant of the Morris contagion model where in period $t = 0$ some nodes play $a = 0$ and others play $a = 1$ (arbitrarily), and subsequently in each period t each node i plays $a = 1$ if and only if at least $q = 0.5$ of its neighbors played $a = 1$ in period $t - 1$. (The difference from the model in lecture is that now nodes can switch from $a = 1$ to $a = 0$ in addition to switching from $a = 0$ to $a = 1$.) Give an example where this process cycles forever.

Solution. (a) At least one of the $m+1$ disjoint sets has no initial node seeded as initial nodes seeded is m. Call one of these sets S. Thus, in period 0 no one in S is taking action 1. If at any point in time t no one in S takes action 1, then at time $t+1$ no one in S will take action 1, by the fact that S is more than $(1-q)$ cohesive. Thus, all nodes in S never take action 1.

(b) Consider the two-node network with one link, where at period 0 node 1 takes action 1 and node 2 takes action 0. This will cycle forever.

Problem 2. In each part of this question, we verbally describe a classic multi-agent decision problem. Formally express each one as a normal form game, and find all (pure and mixed) Nash equilibria.

(a) *Partnership:* Two partners in a firm each decide whether to work or rest. Each partner earns \$100 of profit for the firm if she works, regardless of what the other partner does. All profits earned are divided equally between the two partners. Each partner also derives a private benefit worth \$75 to her from resting. This private benefit cannot be shared with the other partner.

(b) *Stag Hunt:* Two hunters can each hunt stag or hare. If both hunt stag, they catch a stag and get 100 pounds of meat each. If a hunter hunts hare, she catches a hare and get 10 pounds of meat, regardless of what the other hunter does. If a hunter hunts stag while the other hunts hare, the hunter hunting stag catches nothing.

(c) *Chicken:* Two drivers approach each other on a narrow road. Each can either continue or swerve. If one continues and the other swerves, the driver who continues gets a payoff of 1 for appearing brave, and the driver who swerves gets a payoff of 0 for appearing cowardly. If both swerve, both get a payoff of 0. If both continue, they collide and both get a payoff of -10.

(d) *Rock-Paper-Scissors:* Two players each decide whether to throw Rock, Paper, or Scissors. Rock beats Scissors, Scissors beats Paper, and Paper beats Rock. The winner gets \$1 from the loser. No money changes hands in case of a tie.

(e) *Modified Rock-Paper-Scissors:* Same as above, but now the amount of money won/lost is not always \$1. Instead:

Rock *crushes* Scissors: winner gets \$10 from loser when Rock beats Scissors.

Scissors *cut* Paper: winner gets \$5 from loser when Scissors beats Paper.

Paper *covers* Rock: winner gets \$2 from loser when Paper beats Rock.

Ties are treated as in standard Rock-Paper-Scissors.

Solution.

(a) *Partnership:* The only Nash equilibrium is (rest, rest).

	work	rest
work	100,100	50,125
rest	125,50	75,75

(b) *Stag Hunt:*

	stag	hare
stag	100,100	0,10
hare	10,0	10,10

Both (stag, stag) and (hare, hare) are pure-strategy Nash equilibria. There is also a mixed-strategy equilibrium in which each player plays stag with probability p and hare with probability $1-p$. For the players to be willing to mix between the two actions, they must lead to the same expected payoff given the opponent's strategy:

$$100p = 10 \iff p = \frac{1}{10}$$

(c) *Chicken:*

	continue	swerve
continue	-10,-10	1,0
swerve	0,1	0,0

Both (continue, swerve) and (swerve, continue) are pure-strategy Nash equilibria. As before, there is also a mixed-strategy equilibrium where both play continue with probability p. This time

$$(1-p) - 10p = 0 \iff p = \frac{1}{11}.$$

(d) *Rock-Paper-Scissors:*

	rock	paper	scissors
rock	0,0	-1,1	1,-1
paper	1,-1	0,0	-1,1
scissors	-1,1	1,-1	0,0

The only Nash equilibrium is a mixed-strategy equilibrium in which each player chooses each of rock, paper, and scissors with probability $p = 1/3$.

(e) *Modified Rock-Paper-Scissors:*

	rock	paper	scissors
rock	0,0	-2,2	10,-10
paper	2,-2	0,0	-5,5
scissors	-10,10	5,-5	0,0

The only Nash equilibrium is a mixed-strategy equilibrium in which players play rock with probability p, paper with probability q, and scissors with probability $1-p-q$. For players to be indifferent between the three actions, it must be that

$$-2q + 10(1-p-q) = 2p - 5(1-p-q) = -10p + 5q \iff p = \frac{5}{17}, \quad q = \frac{10}{17}.$$

Problem 3. This problem investigates "Downsian political competition" (introduced by Anthony Downs in his book, *An Economic Theory of Democracy*), a major paradigm in political science.

Consider a population of voters uniformly distributed along an ideological spectrum from left ($x = 0$) to right ($x = 1$). Each of the candidates for a single office simultaneously chooses a campagin platform (i.e., a point on the line between $x = 0$ and $x = 1$). The voters observe the candidates' choicees, and then each voter votes for the candidate whose platform is closest to the voter's position in the spectrum.

For example, if there are two candidates and they choose platforms $x_1 = .3$ and $x_2 = .6$, then all voters to the left of $x = .45$ vote for candidate 1 and all those to the right of $x = .45$ vote for candidate 2, so candidate 2 wins with 55% of the vote. Assume that any candidates who choose the same platform equally split the votes cast for that platform, and that ties among the leading vote-getters are resolved by coin flips.

(a) Suppose there are two candidates, and that the candidates soley try to maximize their probability of getting elected (this is called *office-motivated candidates* in political science). Solve for the pure-strategy Nash equilbrium, and prove that it is unique. *(Extra credit: prove that there is no mixed-strategy NE.)*

(b) Suppose there are two candidates, and that now the candidates care soley about the winning platform and not about who wins (this is called *policy-motivated candidates*). Specifically, Candidate 1 is a left-winger, Bernard: if the winning candidate chose platform x, Bernard's payoff is $1 - x$. Similarly, Candidate 2 is a right-winger, Don: if the winning candidate chose platform x, Don's payoff is x. Note that except for the payoffs the game is exactly the same as in part (a). Solve for the unique pure-strategy Nash equilibrium.

(c) Suppose candidates are office-motivated as in part (a), but assume there are now three candidates. Find one pure-strategy NE.

Solution.

(a) In the Nash equilibrium, both politicians choose $x = 0.5$, get half of the vote, and are elected with probability 0.5. There are no other NE. Suppose to the contrary that there is a NE where at least one politician does not choose $x = 0.5$ with probability 1. If there is exactly one such politician, then she wins with probability < 0.5. But if she deviates to $x = 0.5$, then she wins with probability 0.5, which is a contradiction. If instead neither politician chooses $x = 0.5$ with probability 1, then note that one of them wins with probability ≤ 0.5. But if she deviates to $x = 0.5$, then she wins with probability > 0.5, which is a contradiction.

(b) The unique pure-strategy NE is again for both politicians to choose $x = 0.5$. It is easy to see that this is a Nash equilibrium. If Liz deviates to $x \neq 0.5$, then she loses the election for sure and so the implemented policy continues to be 0.5. Therefore, deviating to $x \neq 0.5$ is not profitable for Liz. By a similar argument Don does not have a profitable deviation.

(c) There are many Nash equilibria. Here is one example: player 1 locates at 0.4, player 2 at 0.7, player 3 at 0.8. Player 1 wins with 55% of the vote. Players 2 and 3 do not have profitable deviations: if either of them jumps over player 1, the other one of them wins; anyone sandwiched in between the other two players cannot get more than 20% of the vote, and anyone located to the right of the other two candidates cannot get more than 30% of the vote.

Problem 4. Consider the example of inefficient routing with non-linear latency from Lecture 12; there are two links from origin to destination, with $l_1(x) = x^k$ and $l_2(x) = 1$, where k is a positive number.

(a) Find the socially optimal routing and the equilibrium routing, and calculate the price of anarchy/stability. (Your answers will depend on k.)

(b) Now suppose that the mass of traffic that must be routed from origin to destination is 2 rather than 1 (while the functions $l_1(\cdot)$ and $l_2(\cdot)$ are unchanged). Again, find the socially optimal routing and the equilibrium routing, and calculate the price of anarchy/stability.

(c) Verify that, for any value of k, the total *equilibrium* delay with traffic 1 is less than the total *socially optimal* delay with traffic 2.

In fact, this is a general phenomenon: for any network, the socially optimal routing is worse than the equilibrium routing with half as much traffic. (You do not have to prove this.) Can you think of any implications of this fact for how society can best control traffic?

Solution. (a) The equilibrium is all agents taking link 1. If any agents are on link 2, then the cost of link 1 is $x^k < 1$ and those on link 2 would rather move to link 1.

The socially optimal routing level x is given by

$$\min_{x\in[0,1]} x(x^k) + (1-x)1.$$

The first order condition is $k+1(x^k) - 1 = 0$, which gives $x = \left(\frac{1}{k+1}\right)^{\frac{1}{k}}$. The total cost under this flow x can be written $1 + x(x^k - 1)$ which gives the total cost under the optimum is

$$1 + \left(\frac{1}{k+1}\right)^{\frac{1}{k}}\left(\left(\frac{1}{k+1}\right) - 1\right) = 1 - \frac{k}{k+1}\left(\frac{1}{k+1}\right)^{\frac{1}{k}}.$$

As there is only one equilibrium the price of anarchy and price of stability are the same and equal to

$$\left(1 - \frac{k}{k+1}\left(\frac{1}{k+1}\right)^{\frac{1}{k}}\right)^{-1}.$$

(b) The equilibrium is given by a mass of one going through each of the two routes. This gives everyone a cost of 1, for a total cost of 2 under equilibrium.

The socially optimal routing level x is given by

$$\min_{x \in [0,2]} x(x^k) + (2-x)1$$

which gives the same first order condition as in part (a). Thus, the optimal is again $x = \left(\frac{1}{k+1}\right)^{\frac{1}{k}}$. The overall cost just increases by 1, so we have the total cost is $2 - \frac{k}{k+1}\left(\frac{1}{k+1}\right)^{\frac{1}{k}}$. And the price of anarchy/stability is

$$2\left(2 - \frac{k}{k+1}\left(\frac{1}{k+1}\right)^{\frac{1}{k}}\right)^{-1}.$$

(c) We need to show that

$$1 < 2 - \frac{k}{k+1}\left(\frac{1}{k+1}\right)^{\frac{1}{k}}.$$

Note we have $k > 0$, so $\frac{1}{k+1} < 1$ and $\left(\frac{1}{k+1}\right)^{\frac{1}{k}} < 1$. This gives

$$2 - \frac{k}{k+1}\left(\frac{1}{k+1}\right)^{\frac{1}{k}} > 2 - 1 = 1.$$

This means if we can cut the traffic in half the equilibrium will be better than the original cost. This could be achieved through carpooling or through staggering work times to cut down on total traffic at any time.