Problem 1 (Adoption Curves)**.** Recall that the equation for the adoption curve in the Bass model is

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}$$

where p is the innovation rate and q is the imitation rate.

1. Verify the claim made in lecture that $F(t)$ is concave if $p > q$ and is S-shaped (i.e., convex for sufficiently low t and concave for sufficiently high t) if $p < q$.
2. At what time t is the adoption rate $F'(t)$ the highest? Call this time $t^*(p, q)$.
3. Show that $t^*(p, q)$ is always decreasing in p, but can be increasing or decreasing in q depending on parameters. Explain what are the two opposing forces that lead to the ambiguous dependence of $t^*(p, q)$ on q.

Solution.

1. The second derivative of F is given by

$$F''(t) = \frac{(p+q)^3 e^{-(p+q)t} p \left(q e^{-(p+q)t} - p\right)}{\left(q e^{-(p+q)t} + p\right)^3}.$$

If $p > q$, then $qe^{-(p+q)t} - p$ is negative for all t. If $p < q$, then $qe^{-(p+q)t} - p$ is positive when $t < t_0 := (\log q - \log p)/(p + q)$ and is negative when $t > t_0$.

2. If $p < q$ and the curve is "S"-shaped, the adoption rate is the highest at the time t^* when $F''(t^*) = 0$:

$$t^*(p, q) = \frac{\log q - \log p}{p + q}.$$

which is also t_0 from above. If $p > q$, and the curve is concave, the adoption rate is the highest when $t = 0$.

3. Let's focus on the S-shaped case. The derivative of t^* with respect to p is given by $-1/(p(p+q)) + \log(p/q)/(p+q)^2$, which is always negative since $p < q$. The derivative with respect to q is given by $1/(p(p+q)) + \log(p/q)/(p+q)^2$, which can be negative or positive depending on the values of p and q. In the concave case, of course, t^* is fixed and insensitive to changes in p and q

Problem 2 (Containment in the SIR Model)**.** Consider the SIR model with basic reproduction number R_0. Suppose that at the beginning of the epidemic, the government vaccinates fraction π of the population, which ensures that they never get sick. (That is, vaccinating an individual immediately moves her to the "removed" state.)

1. Write down the dynamic equations and initial conditions for the SIR model as a function of R_0, π, and an initial infected fraction ι.

2. What fraction of the population ever gets sick in the course of the epidemic?

3. Consider the model with $\gamma = 1$, $\pi = 0.5$, and ι arbitrarily small (so you can treat it as zero in the answer to 2). Suppose that a government in this world is choosing a policy to minimize the number of indviduals who get sick over the course of the epidemic. Due to limited funds, the government must choose between two options. The first is increasing testing, which permanently reduces the value of R_0 by 20%. The second is increasing vaccination, which increases the value of π to 0.75. Show numerically for what values of R_0 the government should invest in testing rather than vaccination.

Solution.

1. The dynamic equations are the same as the baseline model — each suspectible person has probability $I(t)$ of interacting with an infected person and conditional probability γR_0 of being infected given that meeting. Thus,

$$\dot{S}(t) = -\gamma R_0 S(t) I(t)$$
$$\dot{I}(t) = \gamma R_0 S(t) I(t) - I(t)$$
$$\dot{R}(t) = \gamma I(t)$$

   The initial conditions, however, are different from the standard model from lecture. They are

$$S(0) = 1 - \iota - \pi$$
$$I(0) = \iota$$
$$R(0) = \pi$$

   which reduce to the case studied in lecture when no one is vaccinated or $\pi = 0$.

2. We combine the first and third differential equation to write

$$\frac{\dot{S}(t)}{S(t)} = -R_0 \dot{R}(t)$$

The solution to this differential equation with initial conditions $S(0) = 1-\iota-\pi$ and $R(0) = \pi$

$$S(t) = (1 - \iota - \pi)e^{-R_0(R(t)-\pi)}$$

Define $S_\infty = \lim_{t\to\infty} S(t)$ and $R_\infty = \lim_{t\to\infty} R(t)$. Observe that $S_\infty = (1 - \iota - \pi)e^{-R_0(R_\infty-\pi)}$. Moreover, since $I_\infty = \lim_{t\to\infty} I(t) = 0$ for the same reason given in the lecture notes, we have that R_∞ is the solution to the following equation

$$R_\infty = 1 - S_\infty = 1 - (1 - \iota - \pi)e^{-R_0(R_\infty-\pi)}$$

We finally note that among the recovered at t, only $R(t) - \pi$ had the disease. Let's call $D(t) = R(t) - \pi$ the number of people at t who actually had the disease, define $D_\infty = R_\infty - \pi$, and then re-write the above as

$$D_\infty = 1 - \pi - (1 - \iota - \pi)e^{-R_0 D_\infty}$$

after subtracting π from both sides. More compactly, as $\iota \downarrow 0$, this is

$$D_\infty = (1 - \pi)(1 - e^{-R_0 D_\infty}) \tag{1}$$

We can now numerically answer our question. For a given value of R_0, we can solve (1) under the baseline with $\pi = 0.5$, the vaccinated scenario where $\pi = 0.75$, and the testing scenario where $R_0 = 0.8R_0$. Note that (1) always has a trivial solution $D_\infty = 0$. If a non-trivial solution exists in $(0, 1)$, that is the relevant long-run limit when a small but positive number of people are initially infected. If a non-trivial solution does not exist, the disease immediately dies out.

Figure 1 shows the results for $R_0 \in [0, 10]$. In this calibration, vaccination always beats testing.

Why? [Note: for credit, you did not have to explain this] One way to think about this is to explore how to right-hand-side of (1) is shifted by changes in R_0 and π. In particular, define

$$f(D_\infty; \pi, R_0) = (1 - \pi)(1 - e^{-R_0 D_\infty})$$

and observe that

$$f_\pi = -(1 - e^{-R_0 D_\infty}) \qquad f_{R_0} = D_\infty(1 - \pi)e^{-R_0 D_\infty}$$

Mechanically, the fixed-point of (1) is going to go down more when this whole f curve shifts down more.[1] Vaccination has a large effect on this curve even when

[1] You could also convince yourself this by implicitly differentiating (1).

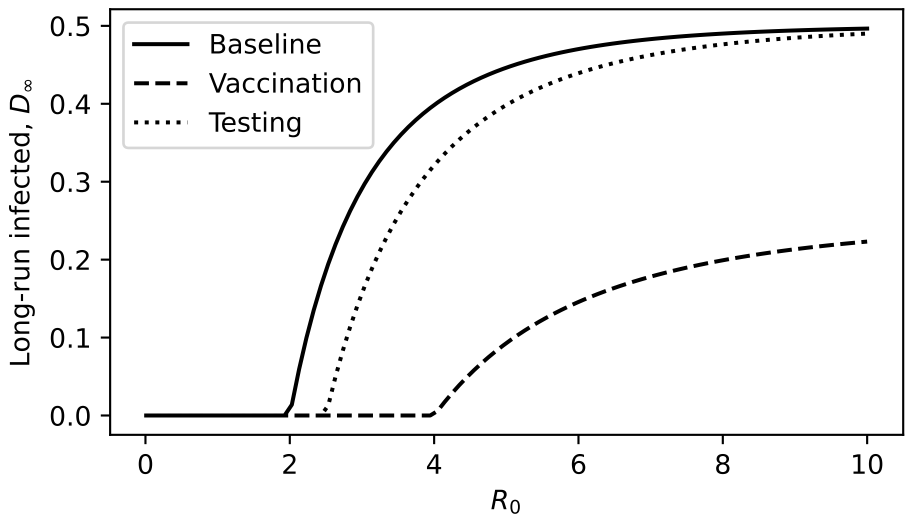

Figure 1: Effectiveness of Policies

D_∞ is very low; while testing only shifts the curve down when D_∞ is relatively high. This makes sense — testing only helps when some people are sick, while vaccination forestalls this completely. That said, this "intuitive explanation" is not a bulletproof policy recommendation — the devil is in the details of just how much a policy can move π or R_0.

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