Problem 1 (Long-Run Consensus)**.** Consider the DeGroot learning model with N agents with initial belief vector $x(0) = (x_1(0), \ldots, x_N(0))$ and an $N \times N$, non-negative, row stochastic matrix T such that, for every period t, we have

$$x(t) = Tx(t-1).$$

(a) Suppose that $N = 3$ and

$$T = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

What properties of this matrix guarantee that, for any initial belief vector $x(0)$, the limit belief $x^* = \lim_{t\to\infty} x(t)$ is well-defined? Compute x^* as a function of $x(0)$.

Solution. The right-stochastic matrix T is aperiodic and strongly connected, so we know from lecture that there is a unique limiting belief x^* that depends only on $x(0)$. Moreover, it is given by $s^\top x(0)$ where the weight vector s solves

$$sT = s \iff s(T - I) = 0$$

and also satisfies $\sum_i s_i = 1$. Solving this linear system of equations by hand, or plugging them into Wolfram Alpha (or a similar tool) gives us $s = (\frac{5}{22}, \frac{8}{22}, \frac{9}{22})$.

(b) Suppose that $N = 6$ and

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 \\ \frac{3}{11} & \frac{4}{11} & \frac{4}{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{4}{13} & \frac{3}{13} & \frac{6}{13} \\ 0 & 0 & 0 & \frac{4}{7} & 0 & \frac{3}{7} \end{pmatrix}.$$

Without doing any computations, does $x^* = \lim_{t\to\infty} x(t)$ exist? Why or why not? If so, which components of the vector x^* will be identical, and which components may differ?

Solution. x^* exists because the network can be broken into two strongly connected components, and the updating matrix is aperiodic. The first three components of x^* will be identical, as will the last three components, but typically the first three components will be different from the last three components.

(c) Prove that, for any N, if there exists an agent i such that $T_{ii} = 1$ and $T_{ji} > 0$ for all $j \neq i$, then $x_j^* \equiv \lim_{t\to\infty} x_j(t)$ is well-defined and equal to $x_i(0)$ for all $j \neq i$.

[Hint: Let $\Delta(t) = \max_{j\in N} |x_i(t) - x_j(t)|$ and let $\underline{T} = \min_{j\neq i} T_{ji}$. Prove that $\Delta(t+1) \leq (1-\underline{T})\Delta(t)$ for all t. Show that this implies that each $x_j(t)$ must converge to $x_i(0)$ as $t \to \infty$.]

Solution. As suggested by the hint, let's define $\underline{T} = \min_k T_{ki} > 0$ and $\Delta(t) = \max_k |x_k(t) - x_i(t)|$. Firstly, notice that since $T_{ii} = 1$ and the rows of T sum to 1, $T_{ij} = 0$ for $j \neq i$ and we must have $x_i(t) = x_i(0)$ for all t by matrix multiplication. Then we can compute

$$\begin{aligned} |x_i(t+1) - x_j(t+1)| &= |x_i(t) - x_j(t+1)| \\ &= |x_i(t) - (Tx(t))_j| \\ &= \; x_i(t) - \sum_{k=1}^{n} T_{jk}x_k(t) \end{aligned}$$

Since $\sum_{k=1}^n T_{jk} = 1$ this can be rewritten as

$$= \; \sum_{k=1}^{n} T_{jk}(x_i(t) - x_k(t))$$

By the triangle inequality $|a+b| \leq |a| + |b|$, this is at most

$$\leq \sum_{k=1}^{n} T_{jk}|x_i(t) - x_k(t)|$$

Since $\sum_{k\neq i} T_{jk} = 1 - T_{ji} \leq 1 - \underline{T}$ and $\Delta(t) = \max_k |x_k(t) - x_i(t)|$, this is at most

$$\leq (1-\underline{T})\Delta(t).$$

Since the above holds for every $j \neq i$, we can deduce that $\Delta(t+1) \leq (1-\underline{T})\Delta(t)$ which means that, since $(1-\underline{T}) < 1$, we must have $\Delta(t) \downarrow 0$ as needed. We conclude that for each k, $x_k(t) \to x_i(0)$ as $t \to \infty$.

Problem 2 (Clustering)**.** Consider the Erdös-Renyi model with $n > 1$ nodes and link probability $p(n)$, which changes as we add nodes. Let $p(n) = \lambda/n$ for all n. Observe that expected degree is held fixed at λ.

(a) Show that as $n \to \infty$ the expected number of triangles in the network converges to $\frac{1}{6}\lambda^3$. (Recall that a *triangle* is a triple of nodes (i, j, k) such that $g_{ij} = g_{ik} = g_{jk} = 1$.) Thus, the expected number of triangles hardly depends on n (once n is large). Explain how this is possible.

Solution. There are $\binom{n}{3}$ possible triangles. $\mathbb{P}(g_{ij} = 1, g_{jk} = 1, g_{ik} = 1) = \mathbb{P}(g_{ij} = 1)\mathbb{P}(g_{jk} = 1)\mathbb{P}(g_{ik} = 1) = \left(\frac{\lambda}{n}\right)^3$

$$\binom{n}{3}\frac{\lambda^3}{n^3} = \frac{n(n-1)(n-2)}{6}\left(\frac{\lambda}{n}\right)^3 \to \frac{\lambda^3}{6} \quad \text{as } n \to \infty.$$

This is constant because the number of triangles is $O(n^3)$ and the probability of all three edges occurring in any triangle is $O\left(\frac{1}{n^3}\right)$.

(b) Show that for large n the expected number of connected triples in the network is approximately $\frac{1}{2}n\lambda^2$. (Recall that a *connected triple* is a triple of nodes (i, j, k) such that $g_{ij} = g_{ik} = 1$.)

Solution. There are $\binom{n}{3}$ possible triples. For any i, j, k let I_{ijk} be an indicator random variable that is 1 if i, j, k are a connected triple and 0 otherwise. Note I_{ijk} takes value 1 with probability $\binom{3}{2}\left(\frac{\lambda}{n}\right)^2$, where $\binom{3}{2}$ comes from needing two of the three edges to be present. So now for the entire graph we have

$$\mathbb{E}\left[\sum_{i,j,k} I_{ijk}\right] = \sum_{i,j,k} \mathbb{E}[I_{ijk}] = \binom{n}{3}\binom{3}{2}\left(\frac{\lambda}{n}\right)^2 \approx \frac{1}{2}n\lambda^2$$

for large n.

(c) Define the *clustering coefficient* for a random network to be the probability that two neighbors of a node are also neighbors of each other. Compute the clustering coefficient for the Erdös-Renyi model with $p(n) = \lambda/n$.

Solution. In the ER model edges are independently realized so the clustering coefficient is $\frac{\lambda}{n}$.

Problem 3 (Phase Transition)**.** Consider again the Erdös-Renyi model with $n > 1$ nodes and link probability $p(n)$. Let A denote the event that node 1 has at least $l \in \mathbb{N}$ neighbhors. Show that there is a phase transition for this event with the threshold function $t(n) = \lambda/n$ for some $\lambda > 0$. [Hint: You may need to use the fact that $\left(1 + \frac{x}{n}\right)^n \approx \exp(x)$ when n is large for any $x \in \mathbb{R}$.]

Bonus: Using a computer language of your choice, code a program that simulates Erdös-Renyi graphs with n nodes and connection probability p. Use a simulation with this program to illustrate the phase transition as $p(n)$ crosses the threshold $t(n) = 1/n$. If you "inspect" the networks produced, what other properties do you notice on either side of the threshold?

Solution. We need to prove that for any $l > 0$:

(i) $\mathbb{P}(A|p(n)) \to 0$ if $\frac{p(n)}{t(n)} \to 0$.

(ii) $\mathbb{P}(A|p(n)) \to 1$ if $\frac{p(n)}{t(n)} \to \infty$.

First assume that $\frac{p(n)}{t(n)} \to 0$. Denote the degree of node 1 by d_1. Since $\frac{p(n)}{t(n)} \to 0$, the expected degree satisfies

$$\mathbb{E}[d_1] = (n-1)p(n) = \frac{p(n)}{t(n)} t(n)(n-1) \approx \frac{p(n)}{t(n)} \frac{r(n-1)}{n}.$$

Therefore, $\mathbb{E}[d_1] \to 0$. This implies that $\mathbb{P}(A|p(n)) \to 0$, since otherwise the expected degree would be strictly positive.

Next assume that $\frac{p(n)}{t(n)} \to \infty$. It follows that $p(n) > \frac{r}{n}$ for sufficiently large n. The probability that A does not occur can bounded as follows:

$$\begin{aligned}
\mathbb{P}(A^c|p(n)) &= \sum_{k=0}^{l-1} \mathbb{P}(d_1 = k|p(n)) = \sum_{k=0}^{l-1} p(n)^k (1-p(n))^{n-1-k} \binom{n-1}{k} \\
&\leq \sum_{k=0}^{l-1} t(n)^k (1-t(n))^{n-1-k} \binom{n-1}{k} \\
&\leq \sum_{k=0}^{l-1} t(n)^k (1-t(n))^{n-1-k} \frac{n^k}{k!} = \sum_{k=0}^{l-1} \left(\frac{r}{n}\right)^k \left(1 - \frac{r}{n}\right)^{n-1-k} \frac{n^k}{k!} \\
&\approx \sum_{k=0}^{l-1} \exp(-r) \frac{r^k}{k!}.
\end{aligned}$$

The second line follows because if the graph was generated using $t(n)$ instead of $p(n)$, each link would be present with smaller probability and hence the probability that node 1 has less than l neighbors (the event A^c) would be larger. Since the above equation is true for any $r \in \mathbb{R}^+$, considering arbitrarily large r, it follows that $\mathbb{P}(A^c|p(n)) \to 0$, or equivalently that $\mathbb{P}(A|p(n)) \to 1$.