Problem 1 (Centrality Measures)**.** Consider each of the following two networks

(a) The directed network represented by

$$g = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) A undirected star network with $N + 1$ nodes

Answer the following questions:

1. Draw each of the networks.

2. Calculate, for each case, the vectors of eigenvector centralty and PageRank for $\alpha = 0.25$ and $\alpha = 0.5$.

3. Comment on how PageRank changes as you increase α. Is this consistent with your intuition that increasing α allocates more centrality to "indirectly" important nodes?

4. In each network, try to compute Katz-Bonacich centrality for $\alpha = 0.5$. For what values of N in the star network (b) is Katz-Bonacich centrality ($\alpha = 0.5$) well-defined?

Hint: if you do not want to invert matrices, try writing out the series expansion $(I - \alpha A)^{-1} = I + \alpha A + \alpha^2 A^2 + \ldots$ in each case.

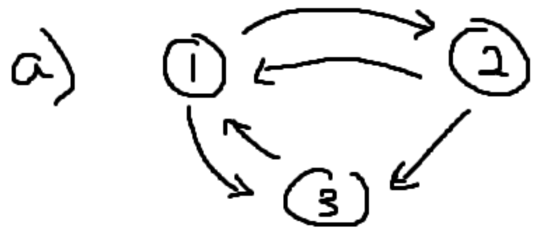
Bonus: use a computer to replicate this exercise for the following matrix:

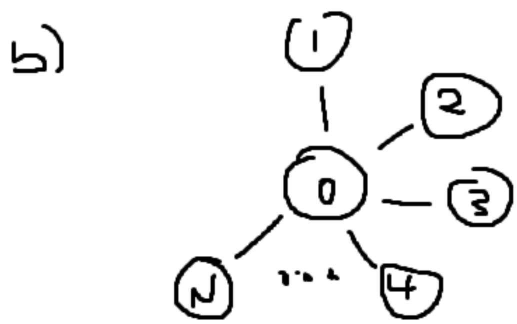
$$g = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Experiment with different values of α for PageRank and Katz-Bonacich centrality. Can you find the upper bound for α such that Katz-Bonacich Centrality remains well-defined? How does this relate to the leading (largest-norm) eigenvalue of g'?

Part 1

Here is an example figure—you could of course label the nodes however you like.





Part 2

Graph (a)

It is convenient to do all the calculations for each graph together. We first calculate **eigenvector centrality**. This is a vector $[v_0, v_1, v_2]'$ that solves

$$\begin{aligned} \lambda v_0 &= v_1 + v_2 \\ \lambda v_1 &= v_0 \\ \lambda v_2 &= v_0 + v_1 \\ 1 &= v_0 + v_1 + v_2 \end{aligned} \tag{1}$$

Combining the first and fourth equations gives $\lambda v_0 = 1 - v_0$ or $v_0 = \frac{1}{1+\lambda}$. Similarly, combining the third and fourth gives $\lambda v_2 = 1 - v_2$ or $v_2 = \frac{1}{1+\lambda}$. Combining these with the second gives $v_1 = \frac{1}{\lambda(1+\lambda)}$.

It remains to solve for λ via the normalization equation. This can be written as

$$1 = \frac{2 + \frac{1}{\lambda}}{1 + \lambda}$$

or

$$\lambda(1 + \lambda) = 2\lambda + 1$$

Solving this quadratic equation, the relevant root is

$$\lambda = \frac{\sqrt{5} + 1}{2}$$

and the vector of eigenvector centrality is therefore

$$v = \begin{bmatrix} \frac{2}{\sqrt{5}+3} & \frac{4}{(\sqrt{5}+3)(\sqrt{5}+1)} & \frac{2}{\sqrt{5}+3} \end{bmatrix}' \tag{2}$$

We next calculate **Katz-Bonacich centrality**. In this solution, we will get the solution by manually inverting the relevant matrix. First, we write.

$$I - \alpha g' = \begin{bmatrix} 1 & -\alpha & -\alpha \\ -\alpha & 1 & 0 \\ -\alpha & -\alpha & 1 \end{bmatrix}$$

We then use the standard formulas for inverting a 3x3 matrix to write

$$(I - \alpha g')^{-1} = \frac{1}{1 - 2\alpha^2 - \alpha^3} \begin{bmatrix} 1 & (\alpha^2 + \alpha) & \alpha \\ \alpha & 1 - \alpha^2 & \alpha^2 \\ \alpha^2 + \alpha & \alpha^2 + \alpha & 1 - \alpha^2 \end{bmatrix}$$

If you did this middle step with the help of Wolfram Alpha, with or without α plugged in, it's totally fine!

Our candidate measure for Katz-Bonacich centrality is therefore

$$v = (I - \alpha g')^{-1} 1' = \frac{1}{1 - 2\alpha^2 - \alpha^3} \begin{bmatrix} 1 + 2\alpha + \alpha^2 \\ 1 + \alpha \\ 1 + 2\alpha + \alpha^2 \end{bmatrix}$$

This vector has all positive entries, and is hence a well-defined centrality measure, if $1 - 2\alpha^2 - \alpha^3 > 0$. This condition holds for $\alpha = 0.25$ and $\alpha = 0.5$. The solutions are

$$v_{0.25} = \begin{bmatrix} \frac{20}{11} & \frac{16}{11} & \frac{20}{11} \end{bmatrix}'$$
$$v_{0.5} = \begin{bmatrix} 6 & 4 & 6 \end{bmatrix}'$$

We will use a similar strategy to calculate **PageRank**. The matrix we want to invert is

$$I - \alpha \tilde{g}' = \begin{bmatrix} 1 & -\alpha/2 & -\alpha \\ -\alpha/2 & 1 & 0 \\ -\alpha/2 & -\alpha/2 & 1 \end{bmatrix}$$

We calculate, again by hand,

$$(I - \alpha \tilde{g}')^{-1} = \frac{1}{1 - \frac{3\alpha^2}{4} - \frac{\alpha^3}{4}} \begin{bmatrix} 1 & \alpha/2 + \alpha^2/2 & \alpha \\ \alpha/2 & 1 - \alpha^2/2 & \alpha^2/2 \\ \alpha^2/4 + \alpha/2 & \alpha^2/4 + \alpha/2 & 1 - \alpha^2/4 \end{bmatrix} \tag{3}$$

and therefore derive the candidate centrality measure,

$$v = \frac{4}{4 - 3\alpha^2 - \alpha^3} \begin{bmatrix} 1 + \frac{3}{2}\alpha + \frac{1}{2}\alpha^2 \\ 1 + \frac{1}{2}\alpha \\ 1 + \alpha + \frac{1}{4}\alpha^2 \end{bmatrix}$$

This is a positive vector, as we desire, for $4 - 3\alpha^2 - \alpha^3 > 0$. But this is clearly satisfied for any $\alpha \in [0, 1)$, as $\alpha^2 < 1$ and $\alpha^3 < 1$. In our cases, we find

$$\begin{aligned} v_{0.25} &= \begin{bmatrix} \frac{40}{27} & \frac{32}{27} & \frac{4}{3} \end{bmatrix}' = \frac{4}{27} \begin{bmatrix} 10 & 8 & 9 \end{bmatrix}' \\ v_{0.5} &= \begin{bmatrix} 12/5 & 8/5 & 2 \end{bmatrix}' = \frac{2}{5} \begin{bmatrix} 6 & 4 & 5 \end{bmatrix}' \end{aligned} \tag{4}$$

Graph (b)

First, **eigenvector centrality**. Let node 0 index the central node, and write the vector as $[v_0, v_1, \ldots, v_N]$. The N equations are

$$\lambda v_0 = \sum_{n=1}^{N} v_i$$
$$\lambda v_i = v_0, \forall 1 \leq n \leq N$$
$$1 = v_0 + \sum_{i=1}^{N} v_i$$

A good guess is that all the $i > 0$ have the same centrality, or $v_i \equiv v_1$. We can then write more concisely[1]

$$\lambda v_0 = N v_1$$
$$\lambda v_1 = v_0$$
$$1 = v_0 + N v_1$$

Combining the first two equations gives $\lambda^2 v_1 = N v_1$ or $\lambda = N^{1/2}$. The first equation then says $N^{1/2} v_0 = N v_1$ or $v_0 = \sqrt{N} v_1$. We finally use the normalization to find

$$v_0 = \frac{\sqrt{N}}{\sqrt{N} + N} \qquad v_1 = \frac{1}{\sqrt{N} + N} \tag{5}$$

Let us next consider **Katz-Bonacich centrality**. Observe the following structure for the matrix $g^k = (g')^k$. For each k, element ij of this matrix gives the number of unique paths between nodes. Notice that all paths have to "bounce between" the center node and the outside nodes. If k is odd, observe that paths have to either start or end at 0, but cannot be both. The number of possible paths is $N^{(k-1)/2}$. Why? There are $k - 2$ nodes that are not fixed (everything except the start or the end), and each could be any of the N non-central nodes. Thus we know

$$g^k = \begin{bmatrix} 0 & N^{(k-1)/2} & N^{(k-1)/2} \ldots \\ N^{(k-1)/2} & 0 & 0 \ldots \\ N^{(k-1)/2} & 0 & 0 \ldots \\ \ldots & & \end{bmatrix} \text{ if } k \text{ is odd}$$

[1] If you are worried about the logical consistency of "guess and check" here, remember the Perron-Frobenius theorem guarantees the existence of a leading eigenvector which is the only one with strictly positive entries.

If instead k is even, then similar logic suggests

$$g^k = \begin{bmatrix} N^{\frac{k}{2}} & 0 & 0\dots \\ 0 & N^{\frac{k}{2}-1} & N^{\frac{k}{2}-1}\dots \\ 0 & N^{\frac{k}{2}-1} & N^{\frac{k}{2}-1}\dots \\ \dots \end{bmatrix} \text{ if } k \text{ is even}$$

Let's now try to calculate Katz-Bonacich centrality. The first element is

$$v_0 = \underbrace{\sum_{k=0}^{\infty}(\alpha^2 N)^k}_{\text{even terms}} + \underbrace{\alpha N \sum_{k=0}^{\infty}(\alpha^2 N)^k}_{\text{odd terms}}$$

if $\alpha^2 N < 1$, then

$$v_0 = \frac{1+\alpha N}{1-\alpha^2 N}$$

Simliarly, the other elements solve

$$v_1 = \underbrace{\sum_{k=0}^{\infty}(\alpha^2 N)^k}_{\text{even terms}} + \underbrace{\alpha \sum_{k=0}^{\infty}(\alpha^2 N)^k}_{\text{odd terms}}$$

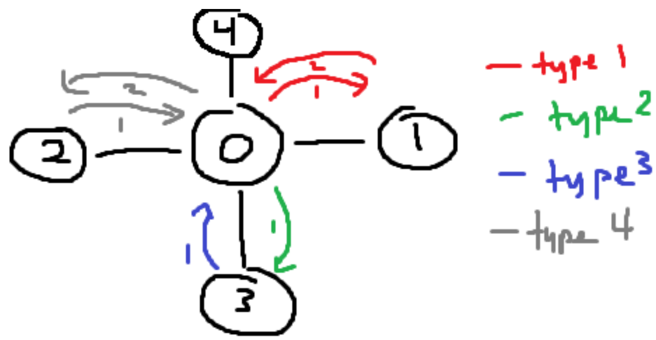
which under the same condition $\alpha^2 N < 1$ is positive and well defined:

$$v_1 = \frac{1+\alpha}{1-\alpha^2 N}$$

To summarize, our solution whenever $\alpha^2 N < 1$ is

$$\begin{aligned} v &= [v_0, v_1, \dots, v_N]' \\ v_0 &= \frac{1+\alpha N}{1-\alpha^2 N} \\ v_i &= \frac{1+\alpha}{1-\alpha^2 N}, \forall i > 0 \end{aligned} \tag{6}$$

Let's finally consider **PageRank**. The calculation here is going to be similar. The normalized adjacency matrix $\tilde{g}$. Let us now use the same logic to calculate its matrix powers. We had four "types" of paths, which are also illustrated (crudely) in a figure with $N = 4$:



1. From 0 to 0 (top left entry of the even powers)
2. From 0 to $\{1, \dots, N\}$ (first column of odd powers)
3. From$\{1, \dots, N\}$ to 0 (first row of odd powers)
4. From $\{1, \dots, N\}$ to $\{1, \dots, N\}$

For 2 and 4, which *end* on the peripheral nodes, our path is split by $1/N$ before being re-combined. For 1 and 3, which end on node 0, the path is split by $1/N$ and recombined N times equally. This logic allows us to conclude

$$g^k = \begin{cases} \begin{bmatrix} 1 & 0 & 0\dots \\ 0 & 1/N & 1/N\dots \\ 0 & 1/N & 1/N\dots \\ \dots \end{bmatrix} & \text{if } k \text{ is even} \\ \begin{bmatrix} 0 & 1 & 1\dots \\ 1/N & 0 & 0\dots \\ 1/N & 0 & 0\dots \\ \dots \end{bmatrix} & \text{if } k \text{ is odd} \end{cases}$$

We now calculate

$$v_0 = \underbrace{\sum_{k=0}^{\infty} (\alpha^2)^k}_{\text{even terms}} + \underbrace{\alpha N \sum_{k=0}^{\infty} (\alpha^2)^k}_{\text{odd terms}}$$

if $\alpha < 1$, then

$$v_0 = \frac{1+\alpha N}{1-\alpha^2}$$

Similarly,

$$v_1 = \underbrace{\sum_{k=0}^{\infty}(\alpha^2)^k}_{\text{even terms}} + \underbrace{\alpha N^{-1}\sum_{k=0}^{\infty}(\alpha^2)^k}_{\text{odd terms}}$$

if $\alpha < 1$, then

$$v_1 = \frac{1+\alpha/N}{1+\alpha N} v_0 = \frac{1+\alpha/N}{1-\alpha^2}$$

To summarize, our solution is

$$\begin{aligned} v &= [v_0, v_1, \ldots, v_N]' \\ v_0 &= \frac{1+\alpha N}{1-\alpha^2} \\ v_i &= \frac{1+\alpha/N}{1-\alpha^2}, \forall i > 0 \end{aligned} \tag{7}$$

Part 3

For graph (b), increasing α puts more (relative) weight on the center node. Here, it was more transparently clear from the calculation itself how the higher-α calculation gives the center node more credit for facilitating indirect links between the peripheral nodes.

For graph (a), increasing α puts more (relative) PageRank weight on nodes 1 and 3 versus node 2. To see the logic behind this, you can look at the matrix powers of g' and observe that nodes 1 and 3 have many more inward walks than node 2.

Part 4

See the calculations for Part 2. The upper bound for N such that Katz-Bonacich(α) is defined is $\bar{N} := \frac{1}{\alpha^2}$. This is all that you have to state for full credit.

The more general property (in a symmetric, irreducible graph), which you might have discovered if you did the bonus problem, is $\alpha < \lambda$ where λ is the largest eigenvalue of g'—you can check this works in the star network, where we calculated the largest eigenvalue as $\sqrt{N}$.

Why? Let's write out the matrix $M \times M$ matrix g' in terms of its eigendecomposition. Because the matrix is real and symmetric, we can write $g' = V \Lambda V'$ and we know the eigenvectors and eigenvalues are real. The sequence expansion of the Leontief inverse matrix, if it exists, is

$$(I - \alpha g')^{-1} = \sum_{k=0}^{\infty} \alpha^k (g')^k \tag{8}$$

Using the eigendecomposition, this is

$$(I - \alpha g')^{-1} = \sum_{k=0}^{\infty} \alpha^k (V \Lambda V')^k \tag{9}$$

Using the fact that $V'V = I$, this becomes

$$(I - \alpha g')^{-1} = V \left(\sum_{k=0}^{\infty} \alpha^k \Lambda^k \right) V' \tag{10}$$

Let's re-write this in terms of the column eigenvectors $(v_i)_{i=1}^M$ and their eigenvalues $(\lambda_i)_{i=1}^M$,

$$(I - \alpha g')^{-1} = \sum_{i=1}^{M} \sum_{k=0}^{\infty} \alpha^k \lambda_i^k v_i v_i' \tag{11}$$

We need each of the geometric sums to converge. By the Perron-Frobenius theorem, we know that there is a unique largest (in norm) eigenvalue which we can index as λ_1. Thus $\alpha < \lambda_1$ is necessary and sufficient for all the sums in () to converge to a well-defined Leontief inverse, and hence we have a well-defined Katz-Bonacich centrality. You can also use , plus a little more reasoning, to discover what Katz-Bonacich centrality limits to as $\alpha \to \lambda_1$—you may have already seen this "coincidence" in your numerical explorations!