**Problem 1.** Consider the model of cooperation on networks: each period, each player $i$ chooses an effort level $x_i \geq 0$ and receives payoff $u_i(x) = \sum_{j \neq i} f(x_j) - x_i$, and each player observes only her neighbors actions. Assume $f$ is the square root function: $f(x_i) = \sqrt{x_i}$.

(a) Suppose the network is an $n$-player clique. Prove that the maximum equilibrium cooperation level for each player is the same number $w > 0$, given by

$$w = \delta (n-1) \sqrt{w}.$$

(b) Suppose the network is a $n+1$-player star. Prove that the maximum equilibrium cooperation level of the center player and the maximum cooperation level of each periphery player are given by $y, z > 0$, respectively, where $y$ and $z$ solve the system of equations

$$\begin{aligned} y &= \delta n \sqrt{z} \\ z &= \delta \sqrt{y} + \delta^2 (n-1) \sqrt{z}. \end{aligned}$$

(c) Let $n = 5$. Numerically, find one discount factor $\delta$ for which $w > y$, and find another discount factor $\delta'$ for which $w < y$. Which is larger, $\delta$ or $\delta'$? Explain intuitively why one discount factor leads to more cooperation in the clique and the other leads to more cooperation in the star.

*Solution.* (a) The maximum cooperation level that can be supported in equilibrium is given by the (component-wise) greatest solution to the system of equations:

$$x_i = (1-\delta) \sum_{t=1}^{\infty} \delta^t \sum_{j \in N_i(t)} f(x_j).$$

In the $n$-player clique network, $N_i(t) = \{j : j \neq i\}$ for all $t \geq 1$. Therefore, the above expression simplifies to

$$x_i = \delta \sum_{j \neq i} f(x_j),$$

which implies that

$$x_i + \delta f(x_i) = \delta \sum_{j=1}^{n} f(x_j).$$

Since $f$ is an increasing function, the above equation implies that $x_i = x_j = w$ for all $i, j$, where $w$ is the solution to

$$w = \delta (n-1) \sqrt{w}.$$

(b)

*Solution.* By an argument similar to the argument from part (a), every periphery player has the same maximum level of cooperation $z$. Since the center player is distance one from periphery players and periphery players are distance two apart from one another, $z$ solves

$$z = \delta \sqrt{y} + \delta^2 (n-1) \sqrt{z}.$$

Since the center player is distance one from every other player, the maximum level of cooperation of the center player solves

$$y = \delta n \sqrt{z}.$$

(c)

*Solution.* When $\delta = 0.1$, then $w = 0.16$, $y = 0.099$, and $z = 0.039$. When $\delta' = 0.5$, then $w = 4$, $y = 4.05$, and $z = 2.6$. $\delta'$ is larger. Intuitively, when $\delta$ is small, the periphery players in the star network have a smaller incentive to cooperate, since they highly discount the cooperation of other players in the periphery. This leads them to have a lower cooperation level, in turn leading to lower cooperation level for the center player. As $\delta \to 1$, the structure of the network becomes irrelevant, because cooperation with distant players is not discounted by much. On the other hand, in the star network, there are $n+1$ players, whereas in the clique there are $n$ players. So players have higher incentives to cooperate on the star network when $\delta$ is large.

*Problem* 2. Alice and Bob are trying to meet for lunch. They can each go to the Cafe or the Diner. Alice's office is near the Cafe, so she knows the exact length of time $w$ it would take to wait in line at the cafe. Bob's office is far from the Cafe, so all he knows is that $w$ is distributed $U[0,2]$. All else equal, Alice would be equally happy eating at the Cafe and the Diner, but Bob prefers eating at the Cafe by an amount $b$ that varies from day to day: assume that Bob knows the exactly value of $b$, while Alice knows only that $b$ is distributed $U[0,3]$, independently of $w$. In addition, Alice and Bob get a benefit of 1 from having lunch together. Summarizing, with Alice as player 1 and Bob as player 2 the payoff matrix is

$$\begin{array}{ccc} & C & D \\ C & 1-w, 1-w+b & -w, 0 \\ D & 0, -w+b & 1, 1 \end{array}$$

(a) Formally model this situation as an incomplete information game.

(b) Find a BNE, and prove that it is unique. How often do Alice and Bob have lunch together?

*Solution.* (a) The set of players is given by $\{A, B\}$. The set of types is given by $\Theta = \Theta_A \times \Theta_B$, where $\Theta_A = [0,2]$ and $\Theta_B = [0,3]$. The players' sets of actions is given by $A_A = A_B = \{C, D\}$. The players' payoffs are given by

$$u_A(a,\theta) = \begin{cases} 1-w & \text{if} \quad a = (C,C), \\ -w & \text{if} \quad a = (C,D), \\ 0 & \text{if} \quad a = (D,C), \\ 1 & \text{if} \quad a = (D,D), \end{cases}$$

where $\theta = (w, b)$, and

$$u_B(a,\theta) = \begin{cases} 1-w+b & \text{if} \quad a = (C,C), \\ 0 & \text{if} \quad a = (C,D), \\ -w+b & \text{if} \quad a = (D,C), \\ 1 & \text{if} \quad a = (D,D). \end{cases}$$

The prior $p$ is given by the uniform distribution over $\Theta$.

(b) We first argue that any BNE must be in threshold strategies. Let $s_i(\theta_i)$ denote the action chosen by type $\theta_i$ of player $i$ in a BNE. The expected payoff to type $w$ of Alice from choosing $C$ is given by

$$\mathbb{P}\left(s_B(\theta_B) = C\right)(1-w) - \mathbb{P}\left(s_B(\theta_B) = D\right) w,$$

whereas her payoff from choosing $D$ is given by

$$\mathbb{P}\left(s_B(\theta_B) = D\right).$$

Since the payoff from $C$ is a strictly decreasing function of $w$ and the payoff from $D$ is independent of $w$, in any BNE, there exists some $w^* \in [0, 2]$ such that Alice chooses $C$ if and only if $w \leq w^*$. The expected payoff to type $b$ of Bob from choosing $C$ is given by

$$\mathbb{P}\left(s_A(\theta_A) = C\right)\left(1 - \mathbb{E}[w|s_A(\theta_A) = C] + b\right) + \mathbb{P}\left(s_B(\theta_B) = D\right)\left(-\mathbb{E}[w|s_A(\theta_A) = D] + b\right),$$

whereas her payoff from choosing $D$ is given by

$$\mathbb{P}\left(s_A(\theta_A) = D\right).$$

The payoff to Bob from $C$ is strictly increasing in $b$ while his payoff from $D$ is independent of $b$. Therefore, there exists some $b^*$ such that Bob chooses $C$ if and only if $b \geq b^*$.

There are several cases to consider. First suppose that $w^* = 2$. Then, Alice always chooses $C$. The payoff to type $b$ of Bob from $C$ is then given by $b$, whereas the payoff from $D$ is given by 0. Therefore, Bob must always choose $C$. This implies that the payoff to type $w$ of Alice from choosing $C$ is given by $1 - w$, and the payoff from choosing $D$ is given by 0. Thus, Alice would choose $C$ if and only if $w \leq 1$, a contradiction.

Next suppose that $w^* = 0$, that is, Alice always chooses $D$. The payoff to type $b$ of Bob from $C$ is then given by $-1 + b$, while the payoff from $D$ is given by 1. So Bob must choose $C$ if an only if $b \geq b^* = 2$. This implies that the payoff to type $w$ of Alice from choosing $C$ is given by $\frac{1}{3}(1 - w) - \frac{2}{3}w = \frac{1}{3} - w$, while her payoff from choosing $D$ is given by $\frac{2}{3} > \frac{1}{3} - w$. This confirms the assumption that $w^* = 0$. Thus, a threshold strategy with $w^* = 0$ and $b^* = 2$ is a BNE. We next argue that this is the only BNE.

Next suppose that $b^* = 0$, that is, Bob always chooses $C$. Then the payoff to type $w$ of Alice from choosing $C$ is given by $1 - w$, whereas her payoff to choosing $D$ is given by 0. Thus, Alice chooses $C$ if and only if $w \leq 1$. The payoff to type $b$ of Bob from choosing $C$ is thus given by $1/2(1/2 + b) + 1/2(-3/2 + b) = -1/2 + b$, whereas his payoff from choosing $D$ is given by $1/2$. Therefore, Bob chooses $C$ if and only if $b \geq 1$, a contradiction.

Next suppose that $b^* = 3$, that is, Bob is always choosing $D$. Then the payoff to type $w$ of Alice from choosing $C$ is given by $-w$, whereas her payoff from choosing $D$ is given by 1. Therefore, Alice will always choose $D$. But then the payoff to type $b$ of Bob from choosing $C$ is given by $-1+b$, whereas his payoff from choosing $D$ is given by 1. Thus, Bob will choose $C$ if and only if $b \geq 2$, a contradiction.

The only remaining case is where $w^* \in (0,2)$ and $b^* \in (0,3)$. If $w^* \in (0,2)$, Alice needs to be indifferent between $C$ and $D$ when $w = w^*$, that is,

$$\mathbb{P}\left(s_B(\theta_B) = C\right) - w^* = \mathbb{P}\left(s_B(\theta_B) = D\right) = 1 - \mathbb{P}\left(s_B(\theta_B) = C\right).$$

This implies that

$$w^* = 2\mathbb{P}\left(s_B(\theta_B) = C\right) - 1 = 2\frac{3-b^*}{3} - 1.$$

Likewise, Bob needs to be indifferent between $C$ and $D$ when $b = b^*$, that is,

$$\frac{w^*}{2}\left(1 - \frac{w^*}{2} + b^*\right) + \frac{2-w^*}{2}\left(-\frac{2+w^*}{2} + b^*\right) = \frac{2-w^*}{2}.$$

Solving the above the equations for $w^*$ and $b^*$, we get $w^* = -1$ and $b^* = 3$, a contradiction.

In the unique BNE, Alice chooses $D$ with probability 1 and Bob chooses $D$ with probability 2/3, so Alice and Bob have lunch together 2/3 of the time.

*Problem* 3. Consider a seller who must sell a single good. There are two potential buyers, each with a valuation for the good that is drawn independently and uniformly from the interval $[0, 1]$. The seller will offer the good using a second-price sealed-bid auction, but he can set a "reserve price" of $r \geq 0$ that modifies the rules of the auction as follows: If both bids are below $r$ then neither bidder obtains the good and it is destroyed. If both bids are at or above $r$ then the regular auction rules prevail. If only one bid is at or above $r$ then that bidder obtains the good and pays $r$ to the seller.

(a) Compute the seller's expected revenue as a function of $r$.

(b) What is the optimal value of $r$ for the seller?

(c) Intuitively, why does the seller benefit from setting a non-zero reserve price?

*Solution.* (a) By an argument identical to the argument in lecture notes, it is weakly dominant for the bidders to bid their true valuations. We next compute the expected revenue of the seller. With probability $r^2$, both valuations are below $r$, and the seller gets zero. With probability $2r(1-r)$, only one valuation is above $r$, and the seller gets $r$. With probability $(1-r)^2$, both valuations are above $r$, and the sellers gets the minimum of the two valuations. The expected value of the minimum of the two valuations conditional on both being above $r$ is equal to the expected value of the minimum of two random variables uniformly distributed over $[r, 1]$. By the formula in the notes, this is exactly equal to $\frac{2r+1}{3}$. Therefore, the expected revenue of the seller is given by

$$0r^2 + r(2r(1-r)) + \frac{2r+1}{3}(1-r)^2 = \frac{1}{3} + r^2 - \frac{4}{3}r^3.$$

(b) Taking the first-order condition with respect to $r$ and setting the derivative equal to zero we get $r = 0$ and $r = 1/2$. Checking the second-order conditions, we find that $r = 0$ is a local minimum while $r = 1/2$ is the global maximum of the function over $[0, 1]$. Therefore, the optimal value of $r$ is equal to $r = 1/2$, leading to an expected revenue of $5/12$ to the seller.

(c) Intuitively, setting a non-zero reserve price is like adding an additional bidder who always bids $r$. Since the expected payoff to the seller is increasing in the number of bidders, adding a bidder increases the payoff to the seller.