You have two hours to complete, scan, and re-upload the exam. The exam is open notes, and all course materials are fair game. However, you are required to work alone.

Problem 1. In this question, assume all networks are undirected.

a. What is the diameter of the ring network (also known as the circle or cycle) with n nodes?

b. What is the diameter of a square lattice with L edges (or $L+1$ nodes) on each side? Next, what is the diameter of a d-dimensional hypercubic lattice with L edges on each side? (This lattice is the graph whose vertices correspond to points $(m_1, \ldots, m_d)$ where $m_j \in \{1, \ldots, L+1\}$ for each dimension $j \in \{1, \ldots, d\}$, and two vertices are linked if and only if their coordinates are the same in $(d-1)$ dimensions and differ by 1 in the remaining dimension.) Hence, what is the diameter of a d-dimensional hypercubic lattice with $n = (L+1)^d$ nodes?

c. Consider a tree network where each node except the leaves (the "end nodes") have k neighbors (assuming that $k \geq 3$). How many nodes can be reached from the root node in d steps? Hence, what is the diameter of the tree as a function of k and the number of nodes, n?

d. Say that a network with n nodes exhibits *small worlds* if its diameter, $D(n)$, satisfies $\lim_{n\to\infty} D(n) / \log n < \infty$. In one or two sentences, explain the real-world phenomenon that this definition is attempting to capture, and explain why the definition corresponds to this phenomenon. Which of the networks in parts (a), (b), and (c) exhibit small worlds, and why?

Problem 2. Consider the Morris contagion model: There is a finite network (N, E). Each node $i \in N$ takes an action $a_i \in \{0, 1\}$. Each node prefers to take action 1 if and only if at least fraction q of their neighbors also take action 1, where $q \in (0, 1)$ is a fixed parameter.

a. Consider a network (N, g) where each node take action $a \in \{0, 1\}$, and action 1 is the optimal action for a node if and only if a fraction of at least q of his or her neighbors take action 1. Show that a sufficient condition for never having a contagion from any group of m nodes is to have at least $m + 1$ disjoint sets of nodes that are each more than $(1 - q)$ cohesive.

b. Consider a variant of the Morris contagion model where in period $t = 0$ some nodes play $a = 0$ and others play $a = 1$ (arbitrarily), and subsequently in each period t each node i plays $a = 1$ if and only if at least $q = 0.5$ of its neighbors played $a = 1$ in period $t - 1$. (The difference from the model in lecture is that now nodes can switch from $a = 1$ to $a = 0$ in addition to switching from $a = 0$ to $a = 1$.) Give an example where this process cycles forever.

Problem 3. Consider an arbitrary network in which, in addition to its position in the network, each node i has a real-valued characteristic $x_i \in \mathbb{R}$. For example, x_i may denote the conversational skills of individual i.

a. Suppose we select a node by first randomly sampling an edge and then randomly sampling one of the nodes on the edge. What is the average value of x_i under this sampling procedure?

b. Now consider the configuration model where each node's degree is drawn independently from a distribution $P(d)$, and then each node's characteristic is draw independently from a distribution $Q(x|d)$. Thus, the characteristic x_i may be correlated with d_i. (For example, individuals with better conversational skills may tend to have more friends.) Suppose we select a node by first choosing an arbitrary node and then randomly sampling a neighbor of that node. What is the expected value of x_i under this sampling procedure in the limit as the number of nodes n goes to ∞?

c. Show that the average value of x_i computed in part (b) is greater than the population average value of x_i if and only if $\text{cov}(x_i, d_i) = \mathbb{E}[x_i d_i] - \mathbb{E}[x_i]\mathbb{E}[d_i]$ is positive.