14.12 Game Theory

Lecture 2: Decision Theory

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Road Map

1. Basic Concepts (Alternatives, preferences,…)
2. Ordinal representation of preferences
3. Cardinal representation – Expected utility theory
4. Modeling preferences in games
5. Applications: Risk sharing and Insurance

Basic Concepts: Alternatives

- Agent chooses between the alternatives
- X = The set of all alternatives
- Alternatives are
  - Mutually exclusive, and
  - Exhaustive

Example

- Options = {Algebra, Biology}
- X = {
- a = Algebra,
- b = Biology,
- ab = Algebra and Biology,
- n = none}

Basic Concepts: Preferences

- A relation $\succcurlyeq$ (on X) is any subset of X×X.
- e.g.,

  $\succcurlyeq^* = \{(a,b),(a,ab),(a,n),(b,ab),(b,n),(n,ab)\}$
- $a \succcurlyeq b \equiv (a,b) \in \succcurlyeq$.
- $\succcurlyeq$ is complete iff $\forall x,y \in X$,

  $x \succcurlyeq y$ or $y \succcurlyeq x$.
- $\succcurlyeq$ is transitive iff $\forall x,y,z \in X$,

  $[x \succcurlyeq y \text{ and } y \succcurlyeq z] \Rightarrow x \succcurlyeq z$.

Preference Relation

Definition: A relation is a preference relation iff it is complete and transitive.

Examples

Define a relation among the students in this class by

- x T y iff x is at least as tall as y;
- x M y iff x's final grade in 14.04 is at least as high as y's final grade;
- x H y iff x and y went to the same high school;
- x Y y iff x is strictly younger than y;
- x S y iff x is as old as y;

More relations

- Strict preference:

$$x \succ y \Leftrightarrow [\, x \succcurlyeq y \text{ and } y \not\succcurlyeq x \,],$$

- Indifference:

$$x \sim y \Leftrightarrow [\, x \succcurlyeq y \text{ and } y \succcurlyeq x \,].$$

Examples

Define a relation among the students in this class by

- x T y iff x is at least as tall as y;
- x Y y iff x is strictly younger than y;
- x S y iff x is as old as y;

Ordinal representation

Definition: $\succcurlyeq$ represented by $u : X \to R$ iff

$x \succcurlyeq y \Leftrightarrow u(x) \geq u(y) \quad \forall x,y \in X.$ (OR)

Example

$\succcurlyeq^{**} =$
$\{(a,b),(a,ab),(a,n),(b,ab),(b,n),(n,ab),(a,a),(b,b),(ab,ab),(n,n)\}$

is represented by u^{**} where

$u^{**}(a) =$

$u^{**}(b) =$

$u^{**}(ab) =$

$u^{**}(n) =$

Exercises

- Imagine a group of students sitting around a round table. Define a relation R, by writing $x\ R\ y$ iff x sits to the right of y. Can you represent R by a utility function?
- Consider a relation $\succcurlyeq$ among positive real numbers represented by u with $u(x) = x^2$.

Can this relation be represented by $u^*(x) = x^{1/2}$?

What about $u^{**}(x) = 1/x$?

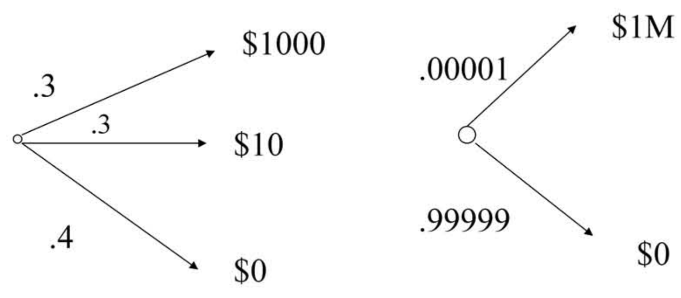
Theorem – Ordinal Representation

Let X be finite (or countable). A relation $\succcurlyeq$ can be represented by a utility function U in the sense of (OR) iff $\succcurlyeq$ is a preference relation.

If $U : X \to R$ represents $\succcurlyeq$, and if $f : R \to R$ is strictly increasing, then $f \circ U$ also represents $\succcurlyeq$.

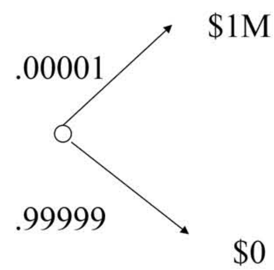
Definition: $\succcurlyeq$ represented by $u : X \to R$ iff
$x \succcurlyeq y \Leftrightarrow u(x) \geq u(y) \quad \forall x, y \in X.$ (OR)

Two Lotteries



Cardinal representation – definitions

- *Z* = a finite set of consequences or prizes.
- A lottery is a probability distribution on *Z*.
- *P* = the set of all lotteries.
- A lottery:



Cardinal representation

- Von Neumann-Morgenstern representation:

Expected value of u under p

A lottery (in P)

$$p \succeq q \Leftrightarrow \underbrace{\sum_{z \in Z} u(z)\,p(z)}_{\mathrm{U}(p)} \geq \underbrace{\sum_{z \in Z} u(z)\,q(z)}_{\mathrm{U}(q)}$$

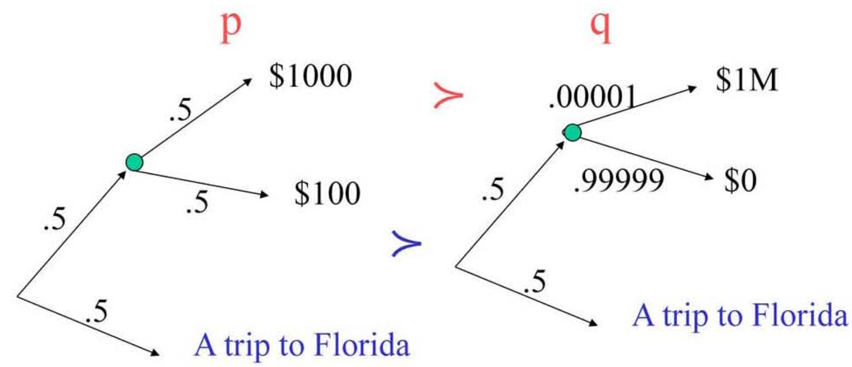
VNM Axioms

Axiom A1: $\succcurlyeq$ is complete and transitive.

VNM Axioms

Axiom A2 (*Independence*): For any p,q,r$\in$P, and any $a \in (0,1]$,

$$ap + (1-a)r \succ aq + (1-a)r \Leftrightarrow p \succ q.$$



VNM Axioms

Axiom A3 (*Continuity*): For any $p,q,r \in P$ with $p \succ q$, there exist $a,b \in (0,1)$ such that

$$a p + (1-a) r \succ q \quad \& \quad p \succ b q + (1-b) r.$$

Theorem – VNM-representation

A relation $\succcurlyeq$ on P can be represented by a VNM utility function $u : Z \rightarrow \mathrm{R}$ iff $\succcurlyeq$ satisfies Axioms A1-A3.

u and v represent $\succcurlyeq$ iff $v = au + b$ for some $a > 0$ and any b.

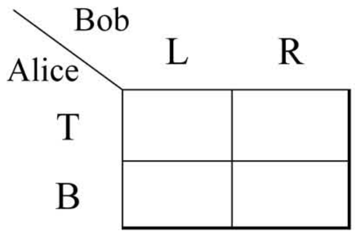
Exercise

- Consider a relation $\succcurlyeq$ among positive real numbers represented by VNM utility function u with $u(x) = x^2$.

  Can this relation be represented by VNM utility function $u^*(x) = x^{1/2}$?

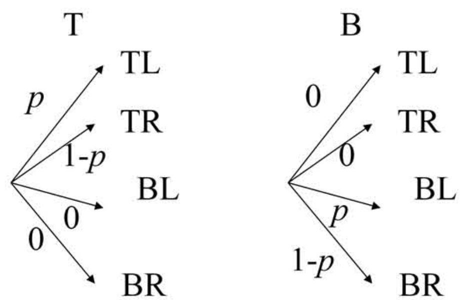
  What about $u^{**}(x) = 1/x$?

Decisions in Games



- Outcomes:

  Z = {TL,TR,BL,BR}
- Players do not know each other's strategy
- p = Pr(L) according to Alice

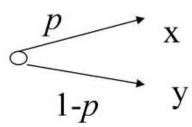


Example

- T $\succsim$ B $\Leftrightarrow p \geq$ ¼; BL ~ BR
- u_A(B,L) = u_A(B,R) = 0
- $p\, u_A$(T,L) + (1-p) u_A(T,R) ≥ 0 $\Leftrightarrow p \geq$ ¼;
- (1/4) u_A(T,L) + (3/4) u_A(T,R) = 0
- Utility of A:

	L	R
T	3	-1
B	0	0

Attitudes towards Risk

- A fair gamble: 

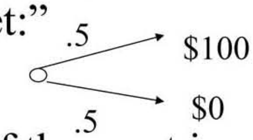
$px+(1-p)y = 0.$

- An agent is *risk neutral* iff
  he is indifferent towards all fair gambles.
- He is *(strictly) risk averse* iff
  he never wants to take any fair gamble.
- He is (strictly) *risk seeking* iff
  he always wants to take fair gambles.

- An agent is risk-neutral iff his utility function is linear, i.e., $u(x) = ax + b$.
- An agent is risk-averse iff his utility function is concave.
- An agent is risk-seeking iff his utility function is convex.

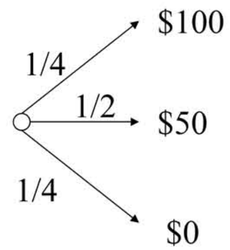
Risk Sharing

- Two agents, each having a utility function u with $u(x) = \sqrt{x}$ and an "asset:"



- For each agent, the value of the asset is 5.
- Assume that the outcomes of assets are independently distributed.

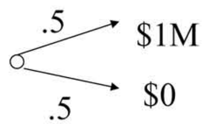
- If they form a mutual fund so that each agent owns half of each asset, each gets



•The Value of the mutual fund for an agent is $(1/4)(100)^{1/2} + (1/2)(50)^{1/2} + (1/4)(0)^{1/2}$
$\approx 10/4 + 7/2 = 6$

Insurance

- We have an agent with $u(x) = x^{1/2}$ and



- And a risk-neutral insurance company with lots of money, selling full insurance for "premium" *P*.

Insurance –continued

- The agent is willing to pay premium P_A where

$$(1M-P_A)^{1/2} \geq (1/2)(1M)^{1/2} + (1/2)(0)^{1/2}$$
$$= 500$$

i.e.,

$$P_A \leq \$1M - \$250K = \$750K.$$

- The company is willing to accept premium

$$P_I \geq (1/2)(1M) = \$500K.$$