Lecture Note 7 – Linking Compensated and Uncompensated Demand: Theory and Evidence

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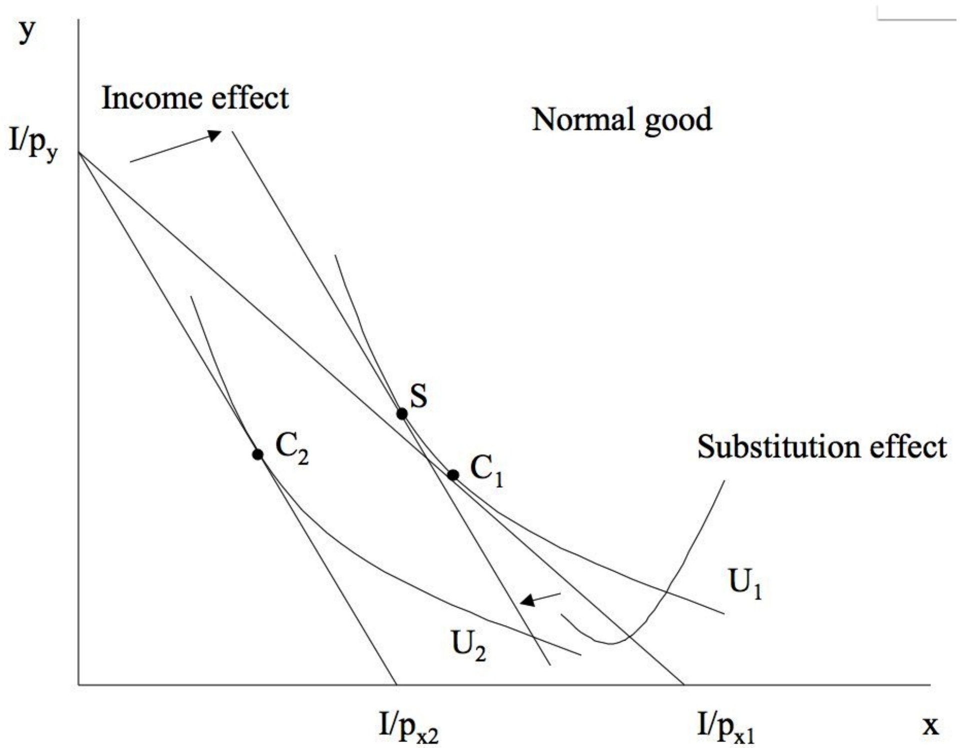
1 Normal, Inferior and "Giffen" Goods

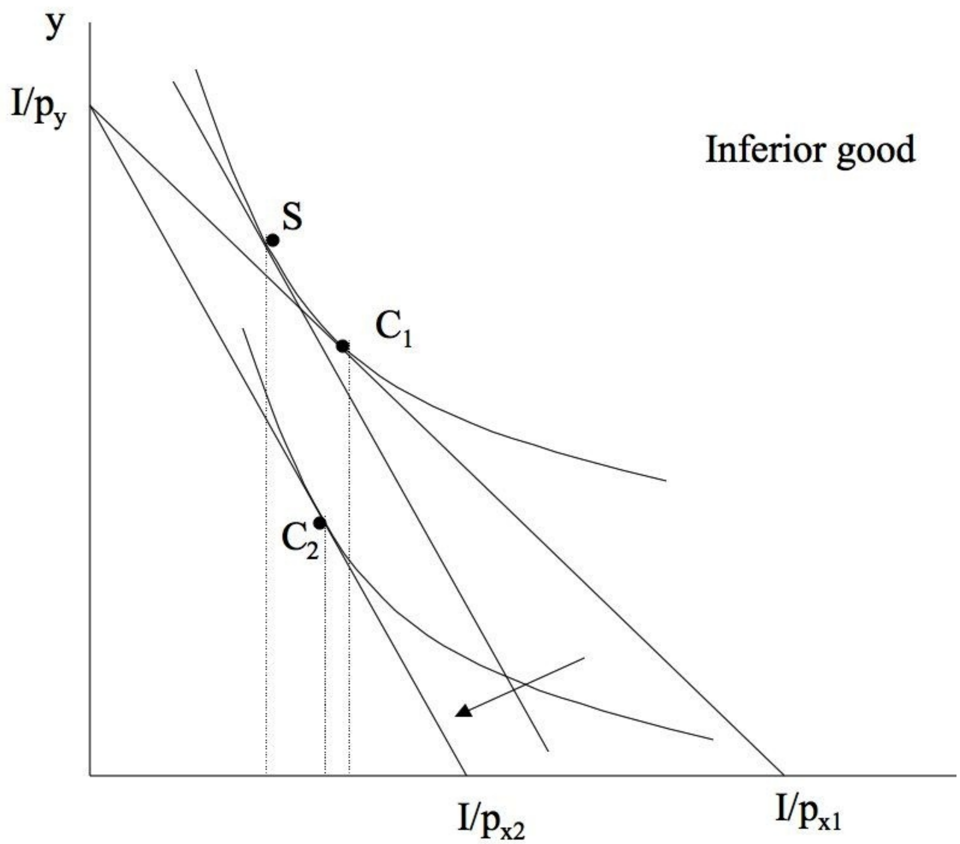
The fact that the substitution effect is always negative but the income effect has an ambiguous sign gives rise to three types of goods:

1. Normal good: $\frac{\partial X}{\partial I} > 0, \frac{\partial X}{\partial p_x}|_{U=U_0} < 0$. For this type of good, a rise in its price and a decline in income have complementary effects—less consumption.



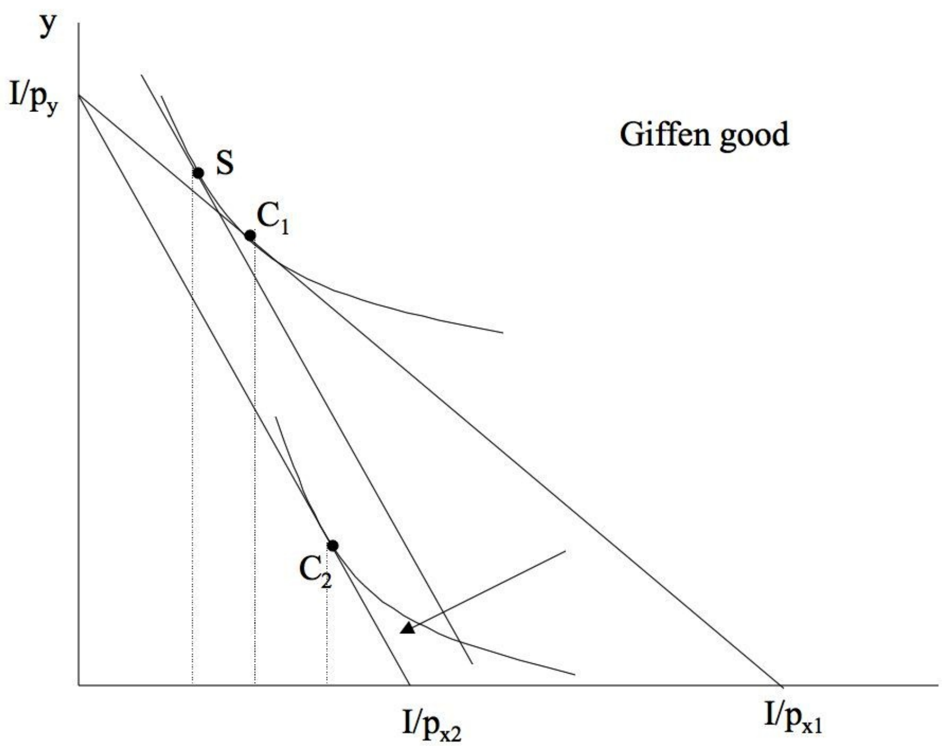
- Although we only observe the movement from C_1 to C_2 (measured along the x-axis), we can conceive of this movement as having two parts: the movement from C_1 to S along the x-axis (substitution effect) and the movement from S to C_2 along the x-axis (income effect).

2. Inferior good: $\frac{\partial X}{\partial I} < 0, \frac{\partial X}{\partial p_x}|_{U=U_0} < 0$. For this type of good, the income and substitution effects are countervailing. Although both derivatives have the same sign, they have opposite effects because a rise in price reduces real income—thereby increasing consumption through the income effect while reducing consumption through the substitution effect.



- Here, the substitution effect is the $S - C_1$ and the income effect is $C_2 - S$ (both along the x-axis).

3. Strongly inferior good ('Giffen' good). $\frac{\partial X}{\partial I} < 0, \frac{\partial X}{\partial p_x}|_{U=U_0} < 0$. Similar to a conventional inferior good, the income and substitution effects are countervailing. But what's special about a Giffen good is that the income effect dominates the substitution effect (in some range): a rise in the price of a Giffen good causes the consumer to buy more of the good (so, demand is effectively upward sloping). Even though a price increase reduces demand due to the substitution effect *holding utility constant*, the consumer is effectively so much poorer due to the income loss that her demand for the inferior good rises.



- The notion of a Giffen good is interesting because it's a non-obvious (in fact, barely plausible) implication of consumer theory; it's hard to imagine a case where when the price of a good rises, demand increases. But theory says such goods could exist. The 2008 Jensen and Miller paper in the *American Economic Review* represents the first rigorous scientific evidence that such goods *do* exist (at least this is the first evidence *for humans*; there is lab evidence of Giffen behavior among some other species). This evidence may speak to the relevance and plausibility of the underlying theory.

- Question: The price of gasoline typically rises during the summer months, as does the gallons of gas consumed per household. Is gas a Giffen good?

1.1 Relationship between Compensated and Uncompensated demand

- These two demand functions—compensated and uncompensated—are quite closely related. But they are not identical.

- Recall from the prior lecture the Expenditure Function

$$E(p_x, p_y, \overline{U}),$$

which is the function that gives the minimum expenditure necessary to obtain utility $\overline{U}$ given prices p_x, p_y.

- For any chosen level of utility $\overline{U}$, the following identity will hold

$$h_x(p_x, p_y, \overline{U}) = d_x(p_x, p_y, E(p_x, p_y, \overline{U})). \tag{1}$$

- In other words, for any chosen level of *utility,* compensated and uncompensated demand must equal to one another. Another way to say this: Fix prices at p_x, p_y. Fix utility at $\bar{U}$. Use the expenditure function to determine the income $\bar{I}$ necessary to attain utility $\bar{U}$ given p_x, p_y. It must be the case that $h_x(p_x, p_y, \overline{U}) = d_x\left(p_x, p_y, \bar{I}\right)$.

- Although these demand curves cross (by construction) at any chosen point, they *do not respond identically to a price change.* In particular differentiating equation (1) with respecting to p_x yields the following equation:

$$\frac{\partial h_x}{\partial p_x} = \frac{\partial d_x}{\partial p_x} + \frac{\partial d_x}{\partial I}\frac{\partial E}{\partial p_x}. \tag{2}$$

Rearranging yields,

$$\frac{\partial d_x}{\partial p_x} = \frac{\partial h_x}{\partial p_x} - \frac{\partial d_x}{\partial I}\frac{\partial E}{\partial p_x}. \tag{3}$$

- In words, the uncompensated demand response to a price change is equal to the compensated demand response $(\partial h_x/\partial p_x)$ minus another term,

$$\frac{\partial d_x}{\partial I}\frac{\partial E}{\partial p_x}.$$

- The $\partial d_x/\partial I$ term should look familiar. It is the income effect on demand for good X. But what is the term multiplying it, $\partial E/\partial p_x$? This term deserves closer inspection.

- Recall the expenditure minimization problem that yields $E(p_x, p_y, \overline{U})$. This problem looks as follows:

$$\min_{X,Y} p_x X + p_y Y \ s.t. U(X, Y) \geq \overline{U}.$$

- The Lagrangian for this problem is:

$$\ell = p_x X + p_y Y \ + \lambda(\overline{U} - U(X, Y)).$$

- The first order conditions for this problem are:

$$\begin{aligned}\frac{\partial \ell}{\partial X} &= p_x - \lambda U_x = 0, \\ \frac{\partial \ell}{\partial y} &= p_y - \lambda U_y = 0, \\ \frac{\partial \ell}{\partial \lambda} &= \overline{U} - U(X, Y).\end{aligned}$$

- The solutions to this problem will have the following Lagrangian multipliers:

$$\lambda = \frac{p_x}{U_x} = \frac{p_y}{U_y}.$$

- And of course, $\overline{U} = U(X^*, Y^*)$ at the optimal choices of X and Y.

- But what is $\partial E/\partial p_x$? In words, holding utility constant, how do optimal expenditures respond to a minute change in the price of one good, X? The answer is:

$$\begin{aligned}\frac{\partial \ell}{\partial P_x} &= X + \left(p_x \frac{\partial X}{\partial p_x} - \lambda U_x \frac{\partial X}{\partial p_x}\right) + \left(p_y \frac{\partial Y}{\partial p_x} - \lambda U_y \frac{\partial Y}{\partial p_x}\right) \\ &= X + \left(p_x \frac{\partial X}{\partial p_x} - p_x \frac{\partial X}{\partial p_x}\right) + \left(p_y \frac{\partial Y}{\partial p_x} - p_y \frac{\partial Y}{\partial p_x}\right) \\ &= X + 0 + 0 \\ &= X.\end{aligned}$$

  Note that we are using the following identities from above:

$$\begin{aligned}p_x &= \lambda U_x \\ p_y &= \lambda U_y.\end{aligned}$$

- So, the short answer is that $\partial E/\partial P_x = X$, and more specifically, $\partial E/\partial P_x = h_x$, where h_x is the Hicksian or compensated demand function.

- This result, called Shephard's Lemma, follows directly from the envelope theorem for constrained problems. Since X and Y are optimally chosen, a minute change in p_x or p_y will not affect the optimal quantity consumed of either good *holding utility constant* (as is always the case with the expenditure function).

- But a price increase will change *total expenditures* — because, to hold utility constant, expenditures must rise when prices rise.

- Since the consumer is already consuming an initial quantity of X_0 units of the good (and hence spending $P_x X_0$ on X), a rise in P_x to $P_x' = P_x + 1$ raises total expenditures needed to maintain the same level of utility by $(P_x' - P_x) X_0 = X_0$.

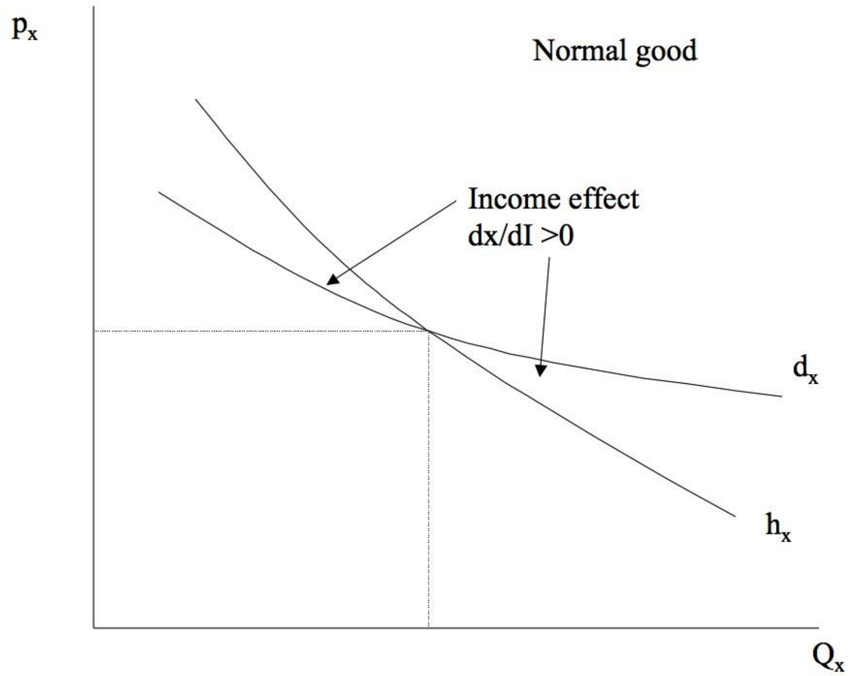
- Concrete example. If you buy 2 cups of coffee a day and the price of coffee rises by 1 cent per cup, how much do we need to compensate you to hold utility constant? To a first approximation, 2 cents (it could never be more, it could actually be less). To hold utility constant given the price change, your expenditures must rise by the price change times the initial level of consumption.

- *Note that this result (Shephard's lemma) holds only locally, i.e., for small price changes.* For a non-negligible price change, the consumer would re-optimize her bundle to re-equate the MRS with the new price ratio. The utility maximization (equivalently, cost minimization) problem is locally flat at the chosen values. Infinitesimal changes in the price ratio therefore have only *second-order* effects on utility and hence do not give rise to first order changes in the consumption bundle.

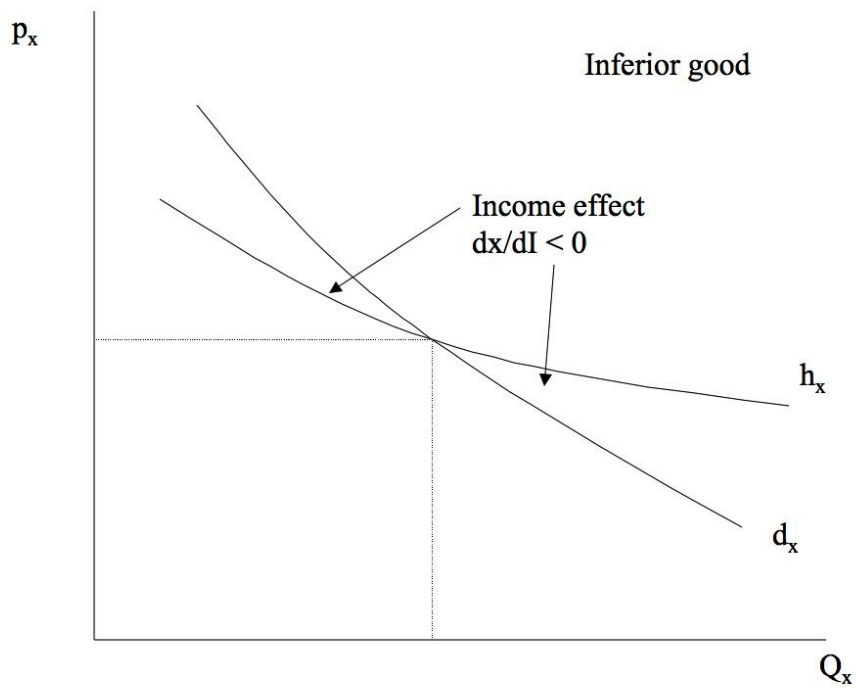
- As noted above, the demand function for X obtained from taking the derivative $\partial E/\partial p_x$ is equal to h_x, the *compensated* demand function, *not* d_x, the uncompensated demand. Why? Because the expenditure function holds utility constant. Hence, any demand function that arises from the expenditure function must also hold utility constant—and so is a compensated demand function.

- So, to reiterate: The derivative of the Expenditure function with respect to the price of a good is the Hicksian (compensated) demand function for that good.

- Graphically the relationship between the compensated and uncompensated demand functions can be seen in the following figures. Recall that the compensated demand curve is composed of only substitution effects, because it represents how consumption changes with the price if the consumer were given enough money to stay on the same indifference curve at each price level. The uncompensated demand includes both the income and substitution effects. Therefore, the difference in slope between the two demand functions is determined by sign and magnitude of the income effects.



Comparison of Compensated (h_x) and Uncompensated (d_x) Demand Curves for a Normal Good



Comparison of Compensated (h_x) and Uncompensated (d_x) Demand Curves for an Inferior Good

1.2 Longer Demonstration of Shephard's Lemma [optional]

Recall the dual of the consumer's problem: minimizing expenditures subject to a utility constraint.

$$
\begin{aligned}
& \min p_x x + p_y y \\
s.t. U(x,y) &\geq v^* \\
\ell &= p_x x + p_y y + \lambda \left(U(x,y) - \bar{U} \right),
\end{aligned}
$$

which gives $E^* = p_x x^* + p_y y^*$ for $U(x^*,\, y^*) = v^*$.

Bear in mind the following first order conditions:

$$\lambda = \frac{p_x}{U_x} = \frac{p_y}{U_y}.$$

Now, calculate

$$\frac{\partial \ell}{\partial p_x} = x + \left(p_x \frac{\partial x}{\partial p_x} - \lambda U_x \frac{\partial x}{\partial p_x} \right) + \left(p_y \frac{\partial y}{\partial p_x} - \lambda U_x \frac{\partial y}{\partial p_x} \right).$$

Substitute in for λ

$$\frac{\partial \ell}{\partial p_x} = x + \left(p_x \frac{\partial x}{\partial p_x} - \frac{p_x}{U_x} U_x \frac{\partial x}{\partial p_x} \right) + \left(p_y \frac{\partial y}{\partial p_x} - \frac{p_y}{U_y} U_y \frac{\partial y}{\partial p_x} \right) = x.$$

Hence,

$$\frac{\partial \ell}{\partial p_x} = \frac{\partial E}{\partial p_x} = x.$$

Note that this x is actually $h_x \left(p_x, p_y, \bar{U} \right)$ since utility is held constant.

1.3 Applying Shephard's lemma

- Returning to equation (3), we can substitute back in using Shephard's Lemma to obtain:

$$\frac{\partial d_x}{\partial p_x} = \frac{\partial h_x}{\partial p_x} - \frac{\partial d_x}{\partial I} \cdot X.$$

- This identity is called the *Slutsky equation*.

- It says that the difference between the uncompensated demand response to a price change (the left-hand side, $\partial d_x / \partial p_x$) is equal to the compensated demand response

$(\partial h_x/\partial p_x)$ minus the income effect scaled by the effective change in income due to the price change (recalling that $X = \partial E/\partial p_x$).

- Notice also the economic content of the final term, $\frac{\partial d_x}{\partial I} \cdot X$. The size of the income effect on total demand for good X in response to a change in p_x depends on the amount of X that the consumer is already purchasing.

- If the consumer is buying large quantities of X, an increase in p_x has a large income effect. If the consumer is consuming zero of good X initially, the income effect of a change in p_x is zero.

- Applying the Slutsky equation to the three types of goods, it's easy to see that:

  - For a normal good ($\frac{\partial d_x}{\partial I} > 0$), the income and substitution effects are complementary.
  - For an inferior good ($\frac{\partial d_x}{\partial I} < 0$), the income and substitution effects are countervailing.
  - For a Giffen good, the income effect dominates: $\left|\frac{\partial d_x}{\partial I} \cdot X\right| > \left|\frac{\partial h_x}{\partial p_x}\right|$ (note that they are both negative.)

- **Effect of rise of p_x in two good economy (X, Y).**

	Uncompensated Demand (Marshallian)	Compensated Demand (Hicksian)
Consumption of X	Substitution: $-$ Income: $+/-$	Substitution: $-$ Income: 0
Consumption of Y	Substitution: $+$ Income: $+/-$	Substitution: $+$ Income: 0
Consumer Utility	$-$	0

1.4 Closing the loop: Uncompensated demand and the indirect utility function.

- One more piece of consumer theory that might come in handy: We concluded directly above that the compensated demand function can be derived just by differentiating the expenditure function. Is there a similar trick for deriving the uncompensated demand function? *Glad you asked!*

- Recall the Lagrangian for the indirect utility function:

$$\begin{aligned}
V &= \max_{x,y} U(X,Y) \ s.t.\ Xp_x + Yp_y \leq I, \\
\ell &= U(X,Y) + \lambda(I - Xp_x - Yp_y), \\
\frac{\partial \ell}{\partial X} &= U_x - \lambda p_x = \frac{\partial \ell}{\partial Y} = U_y - \lambda p_y = \frac{\partial \ell}{\partial \lambda} = I - Xp_x - Yp_y = 0.
\end{aligned}$$

- Now, by the envelope theorem for constrained problems:

$$\frac{\partial \ell}{\partial I} = \frac{\partial V}{\partial I} = \frac{U_y}{p_y} = \frac{U_x}{p_x} = \lambda. \tag{4}$$

The shadow value of additional income is equal to the marginal utility of consumption of either good divided by the cost of the good.

- And by a similar envelope theorem argument:

$$\begin{aligned}
\frac{\partial V}{\partial p_x} &= \frac{\partial \ell}{\partial p_x} = -\lambda X + U_x \frac{\partial X}{\partial p_x} - \lambda p_x \frac{\partial X}{\partial p_x} + U_y \frac{\partial Y}{\partial p_x} - \lambda p_y \frac{\partial Y}{\partial p_x} \quad (5) \\
&= -\lambda X + \left(U_x \frac{\partial X}{\partial p_x} - \lambda p_x \frac{\partial X}{\partial p_x} \right) + \left(U_y \frac{\partial Y}{\partial p_x} - \lambda p_y \frac{\partial Y}{\partial p_x} \right) \quad (6) \\
&= -\lambda X + \left(\lambda p_x \frac{\partial X}{\partial p_x} - \lambda p_x \frac{\partial X}{\partial p_x} \right) + \left(\lambda p_y \frac{\partial Y}{\partial p_x} - \lambda p_y \frac{\partial Y}{\partial p_x} \right) \quad (7) \\
&= -\lambda X \quad (8)
\end{aligned}$$

- Notice the logic of this expression. The utility cost of a one unit price increase in is equal to the additional monetary cost (which is simply equal to X, the amount you are already consuming, times one) multiplied by the shadow value of additional income.

- Returning to the coffee example, a 1 cent price rise costs you 2 cents if you were planning to buy 2 cups. And the value of 2 cents in foregone utility is simply λ times 2 cents.

- Putting together 4 and 9, we get the following expression:

$$-\frac{\partial V(P,I)/\partial P}{\partial V(P,I)/\partial I} = -\frac{-\lambda X}{\lambda} = X = d_x \tag{9}$$

which is called Roy's identity. Notice that we have substituted d_x for X here because we have recovered the Marshallian (uncompensated) demand function. That is, this is the demand for X holding income and other prices constant, not the demand for X holding utility and other prices constant.

- Roy's identity is analogous to Shephard's lemma above; both recover demand functions by differentiating solutions to the consumer's problems with respect to prices. The difference is that by differentiating the expenditure function, Shephard's lemma gives the *compensated* demand function, whereas by differentiating the indirect utility function, Roy's identity gives the *uncompensated* demand function.

- We are now ready to put these tools to work.

2 Giffen goods in China: The Jensen and Miller (2008) Experiment

2.1 Context

- In China in 2005, about 10% of the population survived on less than one dollar per day. (Given China's rapid growth, the contemporaneous number is surely much lower.)

- For this experiment, the sample included 650 households each in Hunan and Gansu provinces (1,300 households and 3,661 individuals).

- Households were selected from the list of "urban poor." Thus, this sample is meant to be representative of the poor population, not the full population.

- Urban poor households by this definition have incomes averaging \$0.41 to \$0.82 per person per day.

- About 90 million Chinese households met this definition at the time of the study.

- The diet among the poor is very simple, consisting mostly of rice and noodles, plus some pork and other meat.

- Most consumers in the sample obtained 70% of total calories from rice or noodles alone.

- Importantly for the study, regional preferences for rice versus noodles vary considerably (Table 1).
  In the South (Hunan), rice is the staple.
  In the North (Gansu), noodles are the staple.

- Meat is generally preferred to rice or noodles, but it is considerably more expensive. Meat typically provides only one-third the calories or protein per Yuan as rice or noodles (Table 2).

2.2 The experiment

- Within each group (Hunan, Gansu), households were randomly assigned to either a control group or one of three treatment groups.
- HH's in the treatment group were given printed vouchers entitling them to prices reductions of 0.10, 0.20 or 0.30 yuan off the price of each $500g$ (1 *jin*) of the staple good (rice or noodles).
- Each treated household received vouchers for 5 months, with the vouchers distributed at the beginning of each month.
- The vouchers were for large quantities, amounting to 750g per person per day for each month of treatment. In practice, this means that households would be very unlikely to use their full quotas. That's important because it means that as far as the household is concerned, the voucher is equivalent to a price reduction in the staple good with no quantity constraint.
- Because the households in this study were extremely poor, they generally only consume the lowest quality variety of the staple good. Thus, substitution to higher qualities (that is, spending more for smaller quantities of higher quality rice) is unlikely to create confounding measurement issues.
- An important subtlety is that Giffen behavior is unlikely to be relevant for households that are either 'too poor' or 'too rich.'
  - Households that are so poor that they are barely meeting their caloric needs are unlikely to be in the range where an increase in effective income due to a decline in the staple price would cause them to buy less of the staple and more of the fancy good. The marginal dollar will still go towards the staple
  - Households that are wealthy enough that they are easily meeting their caloric needs are unlikely to exhibit Giffen behavior because a fall in the price of the staple good does not substantially increase their wealth, and so even if the staple is an inferior good, the income effect will not dominate the substitution effect and so they will tend to buy more of the staple.
  - Households that are poor enough that they are approximately at the subsistence constraint but not substantially above it may exhibit Giffen behavior. For them, a decline in the price of the staple increases effective income so that they may be able to meet their nutritional needs *and* consume some of the fancy good (e.g.,

pork). Thus, a reduction in the staple price may cause them to consume less of the staple and more of the preferred good.

2.3 Data analysis

- One interesting thing about this ‘experiment’ as implemented in the paper is that each household provides its own pre-post comparison over multiple time periods. But there is also a control group. So this is a difference-in-difference design applied to a randomized control trial, which makes the study especially compelling.
- I will let you develop the notation for analyzing this experiment.

2.4 What they find

- The key results for Hunan are found in Table 3 (with additional robustness tests in Table 4).
- Be sure also to study Figure 2. Notice that the percentage change in consumption of the staple is non-monotone in the initial staple calorie share. It’s negative for the wealthier and poorer households in the sample (those that have low and high staple consumption shares) and positive for those with high (60 to 80 percent) but non-corner-solution consumption shares.
- The results for Gansu province are less clear cut. Jensen and Miller discuss at length why that might be.
- What are the major threats to validity in this experiment?
- Are there any alternative interpretations?
- Do these results have any relevance to policy? (Hint: Autor thinks that they do.)