Problem Set 5

1. The "Brusselator" model describes a class of autocatalytic (i.e. self-amplifying) chemical reactions. It reads

$$\dot{x} = a + x^2 y - (b+1)x \tag{1}$$
$$\dot{y} = bx - x^2 y \tag{2}$$

   (a) Find the fixed point(s) x^* and y^*.

   (b) Determine the conditions under which the system undergoes a supercritical Hopf bifurcation.

   (c) Predict the frequency and period of the resulting limit cycle near the bifurcation point.

   (d) Simulate the system numerically[1]. Use the fast Fourier Transform (`fft_demo`) to compute the power spectrum of $x(t) - x^*$ and/or $y(t) - y^*$ near the Hopf bifurcation. Show that your power spectrum contains a peak near the predicted frequency.

2. Figure 1 shows a record of Antarctic temperature fluctuations during the last 800,000 years. These can be inferred by drilling into the ice and looking at how the chemical makeup of the air bubbles changes with depth.

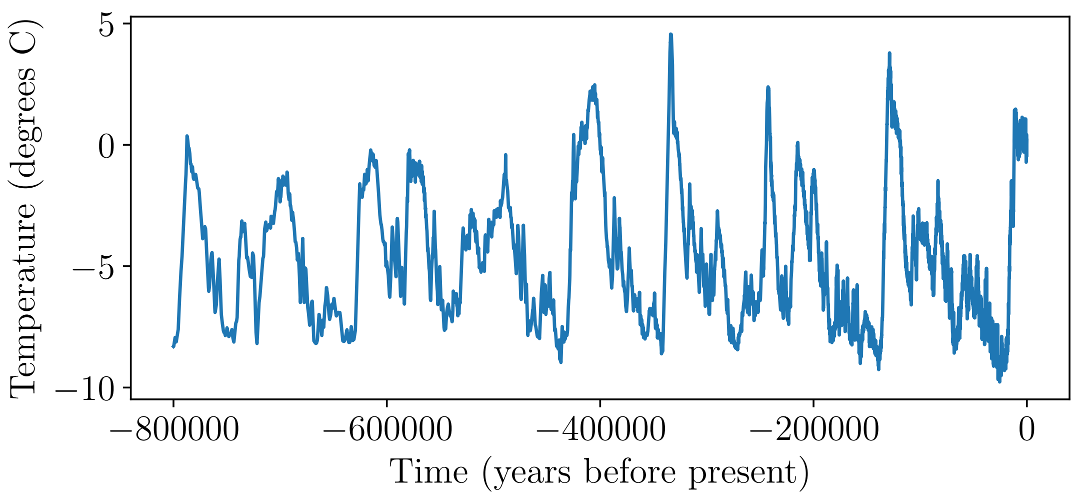

Figure 1: Antarctic temperature fluctuations during the last 800,000 years. Data from Parrenin et al. (2013).

   (a) Use the fast Fourier Transform to compute the power spectrum of this time series (`ice_core_data.csv`)[2], and plot it on log-log axes.

[1] You can use code from past problem sets to help you.

[2] You can import it using numpy.loadtxt or the Matlab "Import Data" feature.

(b) The apparent 100,000 yr periodicity in Figure 1 is synchronized with changes in the eccentricity of Earth's orbit. Other recurring orbital changes include the tilt (obliquity) of Earth's rotation axis, with a period of about 41,000 years, and precession of this axis, with a period of about 23,000 years. Find the peaks corresponding to these frequencies in the power spectrum from (a).

(c) You may notice that the power spectrum behaves roughly like $1/f^{\beta}$, where $\beta \simeq 2$ (a *power law*). We now derive one simple explanation for this.

Suppose you take a *random walk* that starts at $x_0 = 0$. At each increment of time, you take a random step one unit forward or backward with equal probability. Denote the random steps by η_j and assume its mean is zero. Your position at the jth time step relative to time step $j-1$ is

$$x_j = x_{j-1} + \eta_j. \tag{3}$$

We assume that each step η_j is uncorrelated to the others. Now let

$$\hat{x}_k = \sum_{j=0}^{n-1} x_j \exp\left(-i\frac{2\pi jk}{n}\right) \qquad k = 0, 1, \ldots, n-1$$

denote the discrete Fourier transform of the time series x_j. Derive a relation between $\hat{x}_k$ and $\hat{\eta}_k$, the discrete Fourier transform of the white noise sequence η_j. Ignore end effects by assuming $x_{j+n} = x_j$.

(d) Show that the power spectrum

$$\langle |\hat{x}_k|^2 \rangle \propto k^{-2}, \qquad k \ll n/2\pi.$$

(e) If the time series in Figure 1 were the result of purely deterministic processes, why might its power spectrum nevertheless be consistent with that of a random walk?

12.006J/18.353J/2.050J Nonlinear Dynamics: Chaos
Fall 2022