Problem Set 3

1. The Van der Pol equation is

$$\ddot{x} + \mu\dot{x}(x^2 - 1) + x = 0. \qquad (1)$$

Show that $\dot{x} = x = 0$ is a fixed point and analyze its stability.

2. Now we will use `vanderpol` to numerically explore the behavior of the limit cycle in Eq. (1).

(a) Show that for small $\mu \ll 1$ the amplitude of the limit cycle of equation (1) is about 2. What is the period of the limit cycle? Does the period depend on the initial conditions? Contrast this behavior to solutions of the limit case $\mu = 0$. Document your answers with graphs of both the phase space and the time evolution of x.

(b) Show that for large $\mu \gg 1$ the amplitude of the limit cycle is still equal to 2. Note that if you make μ too large you will run into numerical issues. Again, document your answer with graphs of both the phase space and the time evolution. Does the period depend on the initial conditions? For very large μ one can find an approximate period of the van der Pol oscillator (see Strogatz, Section 7.5). Estimate the period of the limit cycle from your computed graphs. How does this period compare with the period derived for large μ in Strogatz?

3. A population Z of zooplankton (microscopic aquatic animals) feeds on a population P of phytoplankton (microscopic aquatic plants).

a) Explain why a plausible model of population dynamics is as follows:

$$\begin{aligned} \frac{dP}{dt} &= aP - bPZ \quad (P \geq 0) \\ \frac{dZ}{dt} &= cPZ - dZ \quad (Z \geq 0) \end{aligned}$$

where a, b, c, d are positive constants.

b) Find the fixed points of the system, determine their stability analytically, and describe the motion in their neighborhood with a sketch in the phase plane. What do your sketches tell you about the evolution of the populations of zooplankton and phytoplankton?

c) Suppose that light intensity from the sun decreases dramatically for a short period of time and kills a quantity of phytoplankton. Discuss qualitatively the different effects this could have on the populations remaining.

d) Confirm your conclusions with numerical solutions (you can modify `vanderpol`).

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