Problem Set 10

1. Consider the logistic map

$$
\begin{aligned}
x_{n+1} &= f(x_n, \mu) && (1)\\
&= 4\mu x_n(1 - x_n) \qquad 0 \le x_n \le 1 && (2)
\end{aligned}
$$

   (a) Let p and q be the points in a 2-cycle; i.e., $p = f(q)$ and $q = f(p)$, with $p \neq q$. Also let $g(x, \mu) = f(f(x, \mu), \mu)$. The 2-cycle is *superstable* when $dg/dx = 0$ at either $x = p$ or $x = q$. Show that superstability requires that either $p = 1/2$ or $q = 1/2$.

   (b) Find the value of μ that yields a superstable 2-cycle.

   (c) Let $\bar{\mu}_n$ be the value of μ for which a 2^n-cycle is superstable. Write an implicit but exact formula for $\bar{\mu}_n$ in terms of the function $f(x, \mu)$.

   (d) Your formula from part 1c can be used to find the μ_n. To help you do this, `superstable` iterates the logistic map for N steps for a range of different values of μ, and plots the final value.

   By choosing appropriate values for the initial condition, N, and range over which to vary μ, find $\bar{\mu}_i$, $i = 2, \ldots, 7$. (Hint: you will need to keep zooming in to smaller and smaller ranges of μ.)

   (e) Evaluate $(\bar{\mu}_6 - \bar{\mu}_5)/(\bar{\mu}_7 - \bar{\mu}_6)$. What is the significance of this number?

2. Another one-dimensional map is given by

$$x_{n+1} = x_n e^{\mu(1-x_n)}, \qquad x \ge 0, \mu \ge 0.$$

Unlike the logistic map, this map has the property that x_n is always positive provided that the initial value is positive. Therefore in an ecological context it has the property that the "population" x_n can never become extinct.

   (a) Find the fixed point(s) and examine their stability.

   (b) Use `bifdiag` to create a rough map of the asymptotic values of x_n for varying μ. The code is set up for the logistic map, so you'll have to make some small changes.

   (c) Find the first four values of μ where period doubling occurs. You may want to identify the approximate region of μ values in which this happens, and re-run `bifdiag` at higher resolution.

   (d) Let μ_n be the value of μ where a period doubling from period 2^n to period 2^{n+1} occurs. The Feigenbaum constant δ may be estimated from the formula

$$\delta = \lim_{n\to\infty} \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}.$$

   Use the values of μ_n obtained above to estimate δ. Is your estimate of δ similar to the value obtained from the logistic map $(4.669\ldots)$. Why or why not?

3. Consider the quartic map

$$x_{n+1} = \mu[1 - (2x_n - 1)^4], \qquad 0 \leq \mu \leq 1$$

As in the previous question, create a rough map of the asymptotic values of x_n for varying μ, find the first few values of μ_n, and estimate δ. Is your estimate of δ similar to the value obtained from the logistic map (4.669...). Why or why not?

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