Lecture notes for 12.006J/18.353J/2.050J, Nonlinear Dynamics: Chaos

D. H. Rothman, MIT
September 16, 2022

Contents

1 Conservation of volume in phase space

Reference: Tolman [1]

We show (via the example of the pendulum) that frictionless systems *conserve* volumes (or areas) in phase space.

Conversely, we shall see, dissipative systems *contract* volumes.

Suppose we have a 3-D phase space, such that

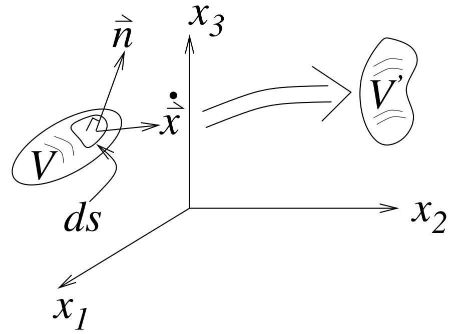
$$\begin{aligned}\dot{x_1} &= f_1(x_1, x_2, x_3)\\ \dot{x_2} &= f_2(x_1, x_2, x_3)\\ \dot{x_3} &= f_3(x_1, x_2, x_3)\end{aligned}$$

or

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}(\vec{x})$$

The equations describe a flow, where $\mathrm{d}\vec{x}/\mathrm{d}t$ is the velocity.

A set of initial conditions enclosed in a volume V flows to another position in phase space, where it occupies a volume V', neither necessarily the same shape nor size:



Assume the volume V has surface S.

Let

- ρ = density of initial conditions in V;
- $\rho\vec{f}$ = rate of flow of points (trajectories emanating from initial conditions) through unit area perpendicular to the direction of flow;
- ds = a small region of S; and
- $\vec{n}$ = the unit normal (outward) to ds.

Then

$$\text{net flux of points out of } S = \int_S (\rho\vec{f}\cdot\vec{n})\mathrm{d}s$$

or

$$\int_V \frac{\partial\rho}{\partial t}\mathrm{d}V = -\int_S (\rho\vec{f}\cdot\vec{n})\mathrm{d}s$$

i.e., a positive flux $\Longrightarrow$ a loss of "mass."

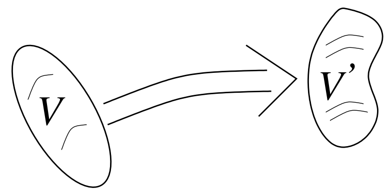
Now we apply the divergence theorem to convert the integral of the vector field $\rho\vec{f}$ on the surface S to a volume integral:

$$\int_V \frac{\partial\rho}{\partial t}\mathrm{d}V = -\int_V [\vec{\nabla}\cdot(\rho\vec{f})]\mathrm{d}V$$

Letting the volume V shrink, we have

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{f})$$

Now follow the motion of V to V' in time δt:



The boundary deforms, but it always contains the same points.

We wish to calculate $\mathrm{d}\rho/\mathrm{d}t$, which is the rate of change of ρ *as the volume moves*:

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial \rho}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{\partial \rho}{\partial x_3}\frac{\mathrm{d}x_3}{\mathrm{d}t} \\
&= -\vec{\nabla}\cdot(\rho\vec{f}\,) \;+\; (\vec{\nabla}\rho)\cdot\vec{f} \\
&= -(\vec{\nabla}\rho)\cdot\vec{f} - \rho\vec{\nabla}\cdot\vec{f} + (\vec{\nabla}\rho)\cdot\vec{f} \\
&= -\rho\vec{\nabla}\cdot\vec{f}.
\end{aligned}
$$

Note that the number of points in V is

$$N = \rho V.$$

Since points are neither created nor destroyed we must have

$$\frac{\mathrm{d}N}{\mathrm{d}t} = V\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\mathrm{d}V}{\mathrm{d}t} = 0.$$

Thus, by our previous result,

$$-\rho V\vec{\nabla}\cdot\vec{f} = -\rho\frac{\mathrm{d}V}{\mathrm{d}t}$$

or

$$\boxed{\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \vec{\nabla}\cdot\vec{f}}$$

This is called the *Lie derivative.* We shall refer to it often in this class.

We next arrive at the following statements by example:

- $\vec{\nabla} \cdot \vec{f} = 0 \Rightarrow$ volumes in phase space are conserved. Characteristic of conservative or Hamiltonian systems.
- $\vec{\nabla} \cdot \vec{f} < 0 \Rightarrow \mathrm{d}V/\mathrm{d}t < 0 \Rightarrow$ volumes in phase space contract. Characteristic of dissipative systems.

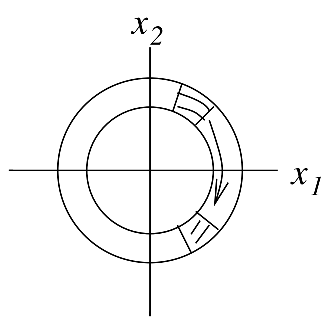
We use the example of the pendulum:

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2) &= x_2 \\ \dot{x_2} &= f_2(x_1, x_2) &= -\frac{g}{l}\sin x_1 \end{aligned}$$

Calculate

$$\vec{\nabla} \cdot \vec{f} = \frac{\partial \dot{x_1}}{\partial x_1} + \frac{\partial \dot{x_2}}{\partial x_2} = 0 + 0$$

Pictorially



Note that the area is conserved.

Conservation of areas holds for *all* conserved systems. This is conventionally derived from Hamiltonian mechanics and the canonical form of equations of motion.

In conservative systems, the conservation of volumes in phase space is known as *Liouville's theorem.*

2 Damped oscillators and dissipative systems

References: Bergé et al. [2], Strogatz [3]

2.1 General remarks

We have seen how conservative systems behave in phase space.
What about dissipative systems?

What is a fundamental difference between dissipative systems and conservative systems, aside from volume contraction and energy dissipation?

- Conservative systems are invariant under time reversal.
- Dissipative systems are not; they are *irreversible*.

Consider again the undamped pendulum:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega^2 \sin\theta = 0.$$

Let $t \to -t$ and thus $\partial/\partial t \to -\partial/\partial t$.
There is no change—the equation is *invariant* under the transformation.

The fact that most systems are dissipative is obvious if we run a movie backwards (ink drop, car crash, cigarette smoke...)

Dissipation therefore must arise in terms proportional to odd time derivatives; i.e., terms that break time-reversal invariance.

In the linear approximation, the damped pendulum equation is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega^2\theta = 0$$

where

$$\begin{aligned} \omega^2 &= g/l \\ \gamma &= \text{damping coefficient} \end{aligned}$$

The sign of γ is chosen so that positive damping opposes motion.

How does the energy evolve over time? As before, we calculate

$$
\begin{aligned}
\text{kinetic energy} &= \frac{1}{2}ml^2\dot{\theta}^2 \\
\text{potential energy} &= mlg(1-\cos\theta) \simeq mlg\left(\frac{\theta^2}{2}\right)
\end{aligned}
$$

where we have assumed $\theta \ll 1$ in the approximation.

Summing the kinetic and potential energies, we have

$$
\begin{aligned}
E(\theta,\dot{\theta}) &= \frac{1}{2}ml^2\left(\dot{\theta}^2 + \frac{g}{l}\theta^2\right) \\
&= \frac{1}{2}ml^2(\dot{\theta}^2 + \omega^2\theta^2)
\end{aligned}
$$

Taking the time derivative,

$$
\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}ml^2(2\dot{\theta}\ddot{\theta} + 2\omega^2\dot{\theta}\theta)
$$

Substituting the damped pendulum equation for $\ddot{\theta}$,

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= ml^2[\dot{\theta}(-\gamma\dot{\theta} - \omega^2\theta) + \omega^2\dot{\theta}\theta] \\
&= -ml^2\gamma\dot{\theta}^2
\end{aligned}
$$

Take $ml^2 = 1$. Then

$$
\frac{\mathrm{d}E}{\mathrm{d}t} = -\gamma\dot{\theta}^2
$$

Conclusion:

$$
\begin{aligned}
\gamma = 0 &\Rightarrow \text{Energy conserved (no friction)} \\
\gamma > 0 &\Rightarrow \text{friction (energy is dissipated)} \\
\gamma < 0 &\Rightarrow \text{energy increases without bound}
\end{aligned}
$$

2.2 Phase portrait of damped pendulum

Express the damped pendulum as

$$
\begin{aligned}
\dot{x} &= \dot{\theta} = y \\
\dot{y} &= \ddot{\theta} = -\gamma\dot{\theta} - \omega^2\sin\theta = -\gamma y - \omega^2\sin x.
\end{aligned}
$$

In the linear approximation, we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The eigenvalues of the system are solutions of

$$(-\lambda)(-\gamma - \lambda) + \omega^2 = 0$$

Thus

$$\lambda = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 4\omega^2}$$

Assume $\gamma^2 \ll \omega^2$ (i.e., weak damping, small enough to allow oscillations). Then the square root is complex, and we may approximate λ as

$$\lambda = -\frac{\gamma}{2} \pm i\omega$$

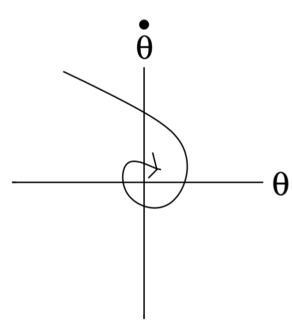
The solutions are therefore exponentially damped oscillations of frequency ω:

$$\theta(t) = \theta_0 e^{-\gamma t/2} \cos(\omega t + \phi)$$

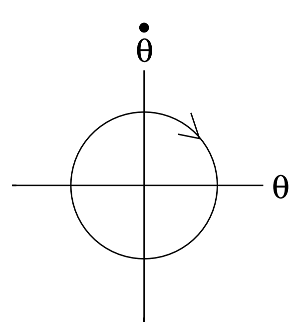
θ_0 and ϕ derive from the initial conditions.

There are three generic cases:

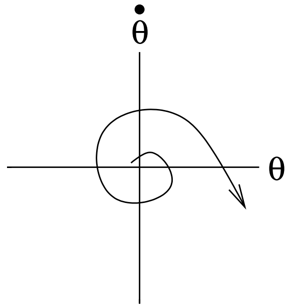
- for $\gamma > 0$, trajectories spiral inwards and are *stable*.



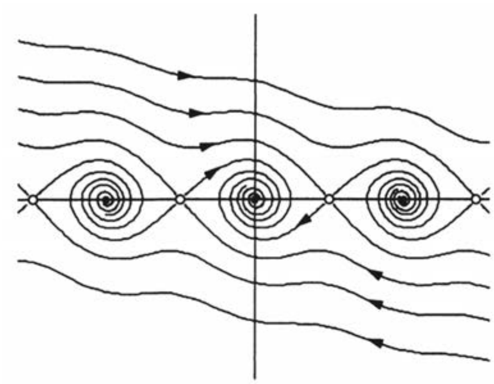
- for $\gamma = 0$, trajectories are *marginally stable* periodic oscillations.



- for $\gamma > 0$, trajectories spiral outwards and are *unstable*.



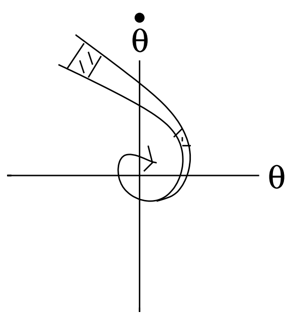
The physical case is the stable case. In the $\theta, \dot{\theta}$-phase plane, the phase portrait looks like this:



© Informa UK Limited. All rights reserved. This content is excluded from our Creative Commons license. For more information, see https://ocw.mit.edu/help/faq-fair-use.

Strogatz [3], Fig. 6.7.7

It is obvious from the phase portraits that the damped pendulum contracts areas in phase space:



We can quantify the contraction of areas using the Lie derivative,

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \vec{\nabla} \cdot \vec{f},$$

which yields

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} \;=\; 0 - \gamma \;=\; -\gamma \;<\; 0$$

The inequality not only establishes area contraction, but γ gives the rate.

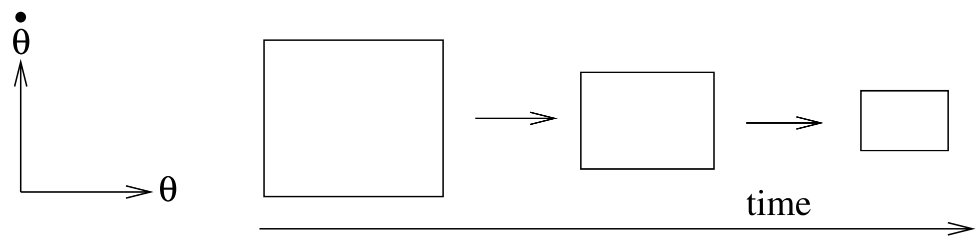
2.3 Summary

Finally, we summarize the characteristics of dissipative systems:

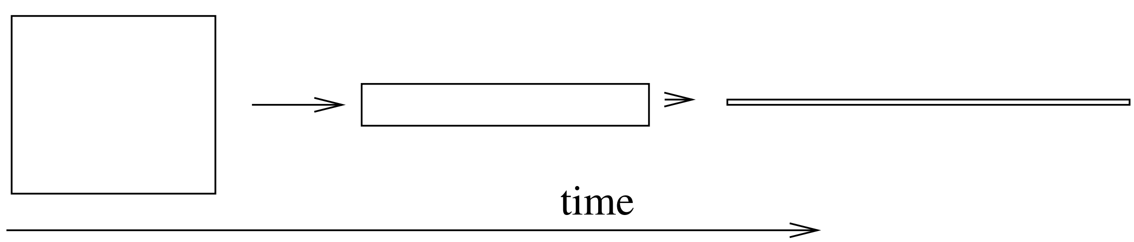
- Energy not conserved.
- Irreversible.
- Contraction of areas (volumes) in phase space.

Note that the contraction of areas is not necessarily simple.

In a 2-D phase space one might expect



However, we can also have



i.e., we can have expansion in one dimension and (a greater) contraction in the other.

In 3-D the stretching and thinning can be even stranger!

References

1. Tolman, R. C. *The Principles of Statistical Mechanics* (Dover, New York, 1979).
2. Bergé, P., Pomeau, Y. & Vidal, C. *Order within Chaos: Towards a Deterministic Approach to Turbulence* (John Wiley and Sons, New York, 1984).
3. Strogatz, S. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* ISBN: 9780429972195 (CRC Press, 2018).

MIT OpenCourseWare
https://ocw.mit.edu

12.006J/18.353J/2.050J Nonlinear Dynamics: Chaos
Fall 2022

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.