(b) Suppose $\mathbf{x}$ differs only slightly from $\mathbf{Q}_k$. Then, $|c_j| \ll |c_k|$ for $j \neq k$. Specialize (a) for this case. *Hint*: Factor out λ_k and c_k^2.

(c) Use (b) to obtain an improved estimate for λ.

$$\mathbf{a} = \begin{bmatrix} 3 & 1 \\ 1 & \frac{3}{2} \end{bmatrix}$$
$$\mathbf{x} \approx \{1, -3\}$$

The exact result is

$$\lambda = 1 \qquad \mathbf{x} = \{1, -2\}$$

3–12. Using Lagrange multipliers, determine the stationary values for the following constrained functions:

(a) $f = x_1^2 - x_2^2$
$g = x_1^2 + x_2 = 0$

(b) $f = x_1^2 + x_2^2 + x_3^2$
$g_1 = x_1 + x_2 + x_3 - 1 = 0$
$g_2 = x_1 - x_2 + 2x_3 + 2 = 0$

3–13. Consider the problem of finding the stationary values of $f = \mathbf{x}^T\mathbf{a}\mathbf{x} = \mathbf{x}^T\mathbf{a}^T\mathbf{x}$ subject to the constraint condition, $\mathbf{x}^T\mathbf{x} = 1$. Using (3–36) we write

$$H = f + \lambda g = \mathbf{x}^T\mathbf{a}\mathbf{x} - \lambda(\mathbf{x}^T\mathbf{x} - 1)$$

(a) Show that the equations defining the stationary points of f are

$$\mathbf{a}\mathbf{x} = \lambda\mathbf{x} \qquad \mathbf{x}^T\mathbf{x} = 1$$

(b) Relate this problem to the characteristic value problem for a symmetrical matrix.

3–14. Suppose $f = \mathbf{x}^T\mathbf{x}$ and $g = 1 - \mathbf{x}^T\mathbf{a}\mathbf{x} = 0$ where $\mathbf{a}^T = \mathbf{a}$. Show that the Euler equations for H have the form

$$\mathbf{a}\mathbf{x} = \frac{1}{\lambda}\mathbf{x} \qquad \mathbf{x}^T\mathbf{a}\mathbf{x} = 1$$

We see that the Lagrange multipliers are the reciprocals of the characteristic values of $\mathbf{a}$. How are the multipliers related to the stationary values of f?

4

Differential Geometry of a Member Element

The geometry of a member element is defined once the curve corresponding to the reference axis and the properties of the normal cross section (such as area, moments of inertia, etc.) are specified. In this chapter, we first discuss the differential geometry of a space curve in considerable detail and then extend the results to a member element. Our primary objective is to introduce the concept of a local reference frame for a member.

4–1. PARAMETRIC REPRESENTATION OF A SPACE CURVE

A curve is defined as the locus of points whose position vector* is a function of a single parameter. We take an orthogonal cartesian reference frame having directions X_1, X_2, and X_3 (see Fig. 4–1). Let $\vec{r}$ be the position vector to a point

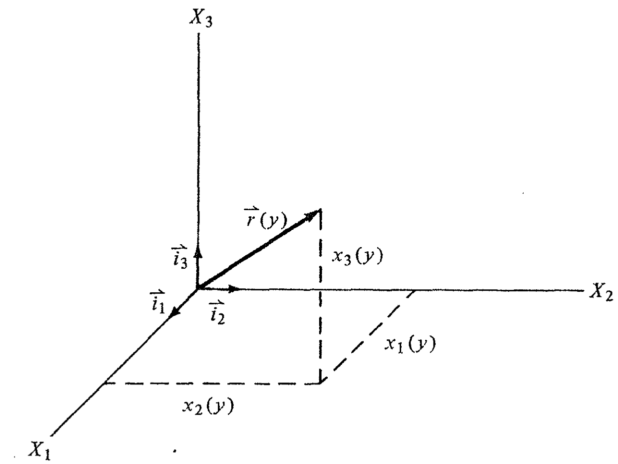


Fig. 4–1. Cartesian reference frame with position vector $\vec{r}(y)$.

* The vector directed from the origin of a fixed reference frame to a point is called the *position vector*. A knowledge of vectors is assumed. For a review, see Ref. 1.

on the curve having coordinates x_j ($j = 1, 2, 3$) and let y be the parameter. We can represent the curve by

$$\vec{r} = \vec{r}(y) \tag{4–1}$$

Since $\vec{r} = \sum_{j=1}^{3} x_j \vec{\imath}_j$, an alternate representation is

$$x_j = x_j(y) \qquad (j = 1, 2, 3) \tag{4–2}$$

Both forms are called the parametric representation of a space curve.

Example 4–1

(1) Consider a circle in the X_1-X_2 plane (Fig. E4–1A). We take y as the polar angle and let a $= |\vec{r}|$. The coordinates are

$$x_1 = a \cos y$$
$$x_2 = a \sin y$$

and

$$\vec{r} = a \cos y \vec{\imath} + a \sin y \vec{\imath}_2$$

(2) Consider the curve (Fig. E4–1B) defined by

$$\begin{aligned} x_1 &= a \cos y \\ x_2 &= b \sin y \\ x_3 &= cy \end{aligned} \tag{4–3}$$

where a, b, c are constants. The projection on the X_1-X_2 plane is an ellipse having semiaxes a and b. The position vector for this curve has the form

$$\vec{r} = a \cos y \vec{\imath}_1 + b \sin y \vec{\imath}_2 + cy \vec{\imath}_3$$

4–2. ARC LENGTH

Figure 4–2 shows two neighboring points, P and Q, corresponding to y and $y + \Delta y$. The cartesian coordinates are x_j and $x_j + \Delta x_j$ ($j = 1, 2, 3$) and the length of the chord from P to Q is given by

$$|\vec{PQ}|^2 = \sum_{j=1}^{3} (\Delta x_j)^2 \tag{a}$$

As $\Delta y \to 0$, the chord length $|\vec{PQ}|$ approaches the arc length, Δs. In the limit,

$$ds^2 = \sum_{j=1}^{3} dx_j^2 \tag{b}$$

Noting that

$$dx_j = \frac{dx_j}{dy} dy \tag{c}$$

we can express ds as

$$ds = +\left[\left(\frac{dx_1}{dy}\right)^2 + \left(\frac{dx_2}{dy}\right)^2 + \left(\frac{dx_3}{dy}\right)^2\right]^{1/2} dy \tag{4–4}$$

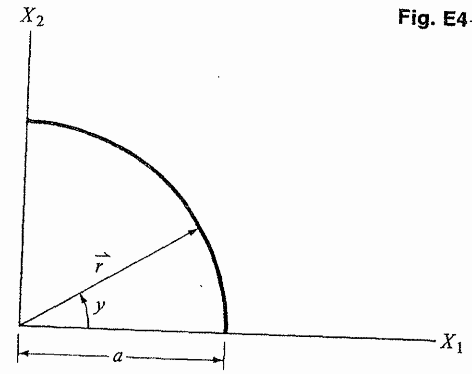


Fig. E4–1A

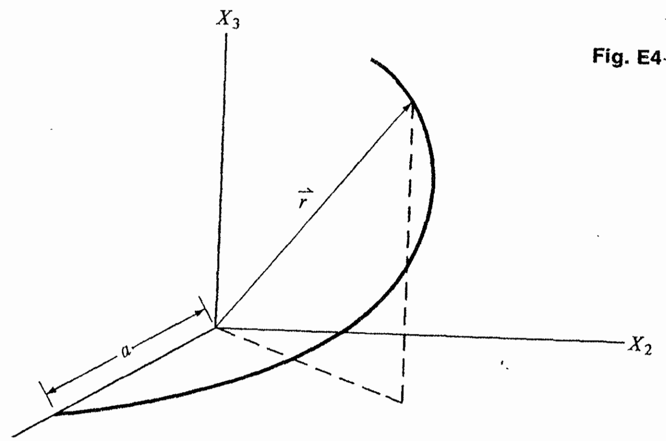


Fig. E4–1B

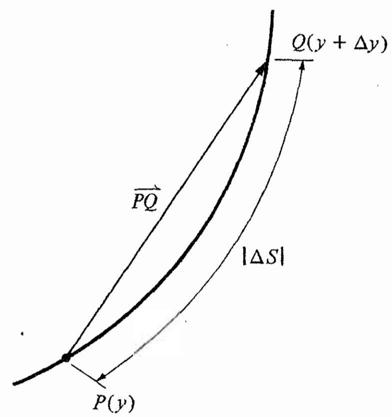


Fig. 4–2. Differential segment of a curve.

Finally, integrating (4–4) leads to

$$s(y) = \int_{y_0}^{y} \left[\left(\frac{dx_1}{dy}\right)^2 + \left(\frac{dx_2}{dy}\right)^2 + \left(\frac{dx_3}{dy}\right)^2 \right]^{1/2} dy \qquad (4\text{–}5)$$

We have defined ds such that s *increases* with *increasing* y. It is customary to call the sense of increasing s the *positive* sense of the curve.

To simplify the expressions, we let

$$\alpha = + \left[\sum_{j=1}^{3} \left(\frac{dx_j}{dy}\right)^2 \right]^{1/2} \qquad (4\text{–}6)$$

Then, the previous equations reduce to

$$ds = \alpha\, dy$$

$$s = \int_{y_0}^{y} \alpha\, dy \qquad (4\text{–}7)$$

One can visualize α as a scale factor which converts dy into ds. Note that $\alpha > 0$. Also, if we take $y = s$, then $\alpha = +1$.

Example 4–2

Consider the curve defined by (4–3). Using (4–6), the scale factor is

$$\alpha = [a^2 \sin^2 y + b^2 \cos^2 y + c^2]^{1/2}$$

We suppose that $b \geqslant a$. One can always orient the axes such that this condition is satisfied. Then, we express α as

$$\alpha = (b^2 + c^2)^{1/2} [1 - k^2 \sin^2 y]^{1/2}$$

where

$$k^2 = \frac{b^2 - a^2}{b^2 + c^2}$$

The arc length is given by

$$s = \int_0^y \alpha\, dy = (b^2 + c^2)^{1/2} \int_0^y [1 - k^2 \sin^2 y]^{1/2}\, dy$$

The integral for s is called an elliptic integral of the second kind and denoted by $E(k, y)$. Then,

$$s = (b^2 + c^2)^{1/2} E(k, y)$$

Tables for $E(k, y)$ as a function of k and y are contained in Ref. 3. When $b = a$, the curve is called a *circular helix* and the relations reduce to

$$\alpha = (a^2 + c^2)^{1/2} = \text{const.}$$

$$s = \alpha y$$

4–3. UNIT TANGENT VECTOR

We consider again the neighboring points, $P(y)$ and $Q(y + \Delta y)$, shown in Figure 4–3. The corresponding position vectors are $\vec{r}(y)$, $\vec{r}(y + \Delta y)$, and

$$\overrightarrow{PQ} = \vec{r}(y + \Delta y) - \vec{r}(y) = \Delta\vec{r} \qquad (a)$$

As $\Delta y \to 0$, $\overrightarrow{PQ}$ approaches the tangent to the curve at P. Then, the unit tangent vector at P is given by*

$$\vec{t} = \lim_{\Delta y \to 0} \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{d\vec{r}}{ds} \qquad (4\text{–}8)$$

Using the chain rule, we can express $\vec{t}$ as

$$\vec{t} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dy}\frac{dy}{ds} = \frac{1}{\alpha}\frac{d\vec{r}}{dy} \qquad (4\text{–}9)$$

Since $\alpha > 0$, $\vec{t}$ always points in the *positive* direction of the curve, that is, in the direction of *increasing* s (or y). It follows that $d\vec{r}/dy$ is also a tangent vector and

$$\left|\frac{d\vec{r}}{dy}\right| = \alpha$$

$$\alpha = \left(\frac{d\vec{r}}{dy} \cdot \frac{d\vec{r}}{dy}\right)^{1/2} \qquad (4\text{–}10)$$

Equation (4–10) reduces to (4–6) when $\vec{r}$ is expressed in terms of cartesian coordinates.

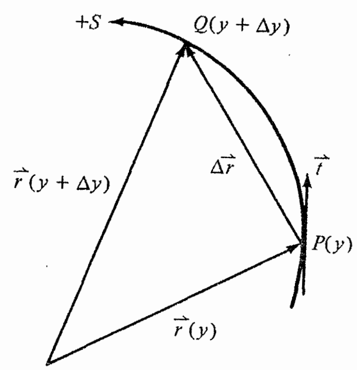


Fig. 4–3. Unit tangent vector at $P(y)$.

* See Ref. 1, p. 401.

Example 4–3

We determine the tangent vector for the curve defined by (4–3). The position vector is

$$\bar{r} = a \cos y\bar{\imath}_1 + b \sin y\bar{\imath}_2 + cy\bar{\imath}_3$$

Differentiating $\bar{r}$ with respect to y,

$$\frac{d\bar{r}}{dy} = -a \sin y\bar{\imath}_1 + b \cos y\bar{\imath}_2 + c\bar{\imath}_3$$

and using (4–9) and (4–10), we obtain

$$\alpha = +[a^2 \sin^2 y + b^2 \cos^2 y + c^2]^{1/2}$$

$$\bar{t} = \frac{1}{\alpha}[-a \sin y\bar{\imath}_1 + b \cos y\bar{\imath}_2 + c\bar{\imath}_3]$$

When $a = b$, $\alpha = [a^2 + c^2]^{1/2} = \text{const}$, and the angle between the tangent and the X_3 direction is constant. A space curve having the property that the angle between the tangent and a fixed direction (X_3 direction for this example) is constant is called a helix.*

4–4. PRINCIPAL NORMAL AND BINORMAL VECTORS

Differentiating $\bar{t} \cdot \bar{t} = 1$ with respect to y, we have

$$\bar{t} \cdot \frac{d\bar{t}}{dy} = 0 \quad \text{(a)}$$

It follows from (a) that $d\bar{t}/dy$ is *orthogonal* to $\bar{t}$. The unit vector pointing in the direction of $d\bar{t}/dy$ is called the *principal normal vector* and is usually denoted by $\bar{n}$.

$$\bar{n} = \frac{1}{\left|\frac{d\bar{t}}{dy}\right|}\frac{d\bar{t}}{dy}$$

where

$$\frac{d\bar{t}}{dy} = \frac{d}{dy}\left(\frac{1}{\alpha}\frac{d\bar{r}}{dy}\right) \quad \text{(4–11)}$$

The binormal vector, $\bar{b}$, is defined by

$$\bar{b} = \bar{t} \times \bar{n} \quad \text{(4–12)}$$

We see that $\bar{b}$ is also a unit vector and the three vectors, $\bar{t}$, $\bar{n}$, $\bar{b}$ comprise a right-handed mutually orthogonal system of unit vectors at a point on the curve. Note that the vectors are *uniquely* defined once $\bar{r}(y)$ is specified. The frame associated with $\bar{t}$, $\bar{b}$ and $\bar{n}$ is called the *moving trihedron* and the planes determined by ($\bar{t}$, $\bar{n}$), ($\bar{n}$, $\bar{b}$) and ($\bar{b}$, $\bar{t}$) are referred to as the *osculating normal*, and *rectifying* planes (see Fig. 4–4).

* See Ref. 4, Chap. 1.

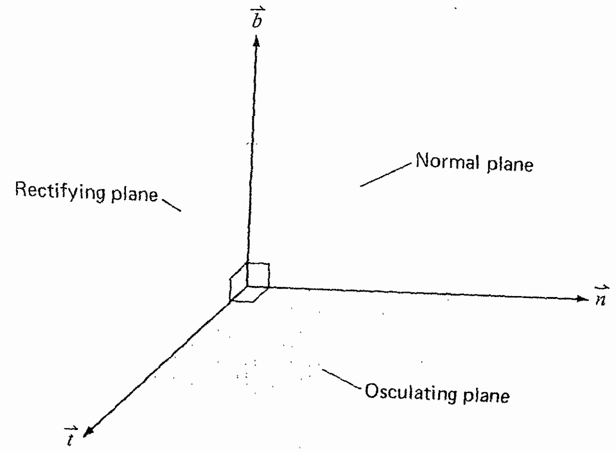


Fig. 4–4. Definition of local planes.

Example 4–4

We determine $\bar{n}$ and $\bar{b}$ for the circular helix. We have already found that

$$\alpha = [a^2 + c^2]^{1/2}$$

and

$$\bar{t} = \frac{1}{\alpha}[-a \sin y\bar{\imath}_1 + a \cos y\bar{\imath}_2 + c\bar{\imath}_3]$$

Differentiating $\bar{t}$ with respect to y, we obtain

$$\frac{d\bar{t}}{dy} = -\frac{a}{\alpha}[\cos y\bar{\imath}_1 + \sin y\bar{\imath}_2]$$

Then,

$$\bar{n} = \frac{1}{\left|\frac{d\bar{t}}{dy}\right|}\frac{d\bar{t}}{dy} = -\cos y\bar{\imath}_1 - \sin y\bar{\imath}_2$$

The principal normal vector is parallel to the X_1-X_2 plane and points in the *inward* radial direction. It follows that the rectifying plane is orthogonal to the X_1-X_2 plane. We can determine $\bar{b}$ using the expansion for the vector product.

$$\bar{b} = \bar{t} \times \bar{n}$$

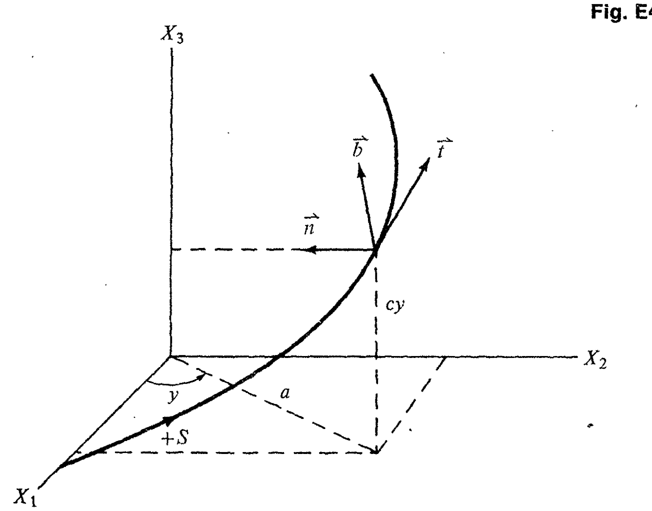
$$= \frac{1}{\alpha}\begin{vmatrix} \bar{\imath}_1 & \bar{\imath}_2 & \bar{\imath}_3 \\ -a \sin y & a \cos y & c \\ -\cos y & -\sin y & 0 \end{vmatrix}$$

This reduces to

$$\bar{b} = \frac{c}{\alpha}\sin y\bar{\imath}_1 - \frac{c}{\alpha}\cos y\bar{\imath}_2 + \frac{a}{\alpha}\bar{\imath}_3$$

The unit vectors are shown in Fig. E4–4.

Fig. E4-4



4-5. CURVATURE, TORSION, AND THE FRENET EQUATIONS

The derivative of the tangent vector with respect to arc length is called the curvature vector, $\bar{K}$.

$$\bar{K} = \frac{d\bar{t}}{ds} = \frac{d^2\bar{r}}{ds^2}$$

$$K = \left|\frac{d^2\bar{r}}{ds^2}\right| = \frac{1}{\alpha}\left|\frac{d\bar{t}}{dy}\right| \tag{4-13}$$

Using (4-11), we can write

$$\bar{K} = \frac{d\bar{t}}{ds} = K\bar{n} \tag{4-14}$$

Note that $\bar{K}$ points in the same direction as $\bar{n}$ since we have taken $K \geqslant 0$. The curvature has the dimension L^{-1} and is a measure of the variation of the tangent vector with arc length.

We let R be the reciprocal of the curvature:

$$R = K^{-1} \tag{4-15}$$

In the case of a plane curve, R is the radius of the circle passing through three consecutive points* on the curve, and $K = |d\theta/ds|$ where θ is the angle between $\bar{t}$ and $\bar{i}_1$. To show this, we express $\bar{t}$ in terms of θ and then differentiate with respect to s. From Fig. 4-5, we have

$$\bar{t} = \cos\theta\bar{i}_1 + \sin\theta\bar{i}_2$$

* See Ref. 4, p. 14, for a discussion of the terminology "three consecutive points."

Then

$$\bar{K} = [-\sin\theta\bar{i}_1 + \cos\theta\bar{i}_2]\frac{d\theta}{ds}$$

and

$$K = \left|\frac{d\theta}{ds}\right| = \frac{1}{R}$$

$$\bar{n} = \frac{d\theta/ds}{|d\theta/ds|}[-\sin\theta\bar{i}_1 + \cos\theta\bar{i}_2]$$

In the case of a space curve, the tangents at two consecutive points, say P and Q, are in the osculating plane at P, that is, the plane determined by $\bar{t}$ and $\bar{n}$ at P. We can interpret R as the radius of the osculating circle at P. It should be noted that the osculating plane will generally vary along the curve.

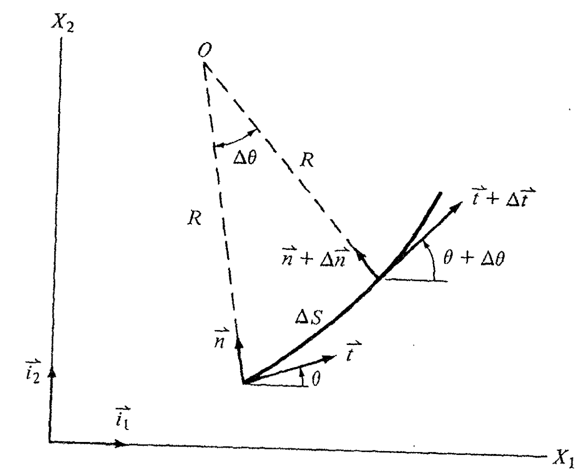


Fig. 4-5. Radius of curvature for a plane curve.

The binormal vector is normal to both $\bar{t}$ and $\bar{n}$ and therefore is normal to the osculating plane. A measure of the variation of the osculating plane is given by $d\bar{b}/ds$. Since $\bar{b}$ is a unit vector, $d\bar{b}/ds$ is orthogonal to $\bar{b}$. To determine whether $d\bar{b}/ds$ involves $\bar{t}$, we differentiate the orthogonality condition $\bar{t}\cdot\bar{b} = 0$, with respect to s.

$$\bar{t}\cdot\frac{d\bar{b}}{ds} = -\bar{b}\cdot\frac{d\bar{t}}{ds}$$

But $d\bar{t}/ds = K\bar{n}$ and $\bar{b}\cdot\bar{n} = 0$. Then, $d\bar{b}/ds$ is also orthogonal to $\bar{t}$ and involves only $\bar{n}$. We express $d\bar{b}/ds$ as

$$\frac{d\bar{b}}{ds} = -\tau\bar{n} \tag{4-16}$$

where τ is called the torsion and has the dimension, L^{-1}.

It remains to develop an expression for τ. Now, $\vec{b}$ is defined by

$$\vec{b} = \vec{t} \times \vec{n}$$

Differentiating with respect to s, we have

$$\frac{d\vec{b}}{ds} = \frac{d\vec{t}}{ds} \times \vec{n} + \vec{t} \times \frac{d\vec{n}}{ds}$$

This reduces to

$$\frac{d\vec{b}}{ds} = \vec{t} \times \frac{d\vec{n}}{ds}$$

since $\vec{n} \times \vec{n} = 0$. Finally, using (4–16), the torsion is given by

$$\tau = -\vec{n} \cdot \vec{t} \times \frac{d\vec{n}}{ds} = \frac{1}{\alpha} \vec{b} \cdot \frac{d\vec{n}}{dy} \qquad (4\text{–}17)$$

Note that τ can be positive or negative whereas K is always positive, according to our definition. The torsion is *zero* for a *plane* curve since the osculating plane coincides with the plane of the curve and $\vec{b}$ is constant.

Example 4–5

The unit vectors for a circular helix are

$$\vec{t} = \frac{1}{\alpha}[-a \sin y\vec{\imath}_1 + a \cos y\vec{\imath}_2 + c\vec{\imath}_3]$$

$$\vec{n} = -\cos y\vec{\imath}_1 - \sin y\vec{\imath}_2$$

$$\vec{b} = \frac{1}{\alpha}[c \sin y\vec{\imath}_1 - c \cos y\vec{\imath}_2 + a\vec{\imath}_3]$$

where

$$\alpha = (a^2 + c^2)^{1/2}$$

Then,

$$K = \frac{1}{\alpha}\left|\frac{d\vec{t}}{dy}\right| = \frac{a}{\alpha^2} = \frac{a}{a^2 + c^2} = \text{const}$$

and

$$\tau = \frac{1}{\alpha}\vec{b} \cdot \frac{d\vec{n}}{dy} = \frac{c}{\alpha^2} = \frac{c}{a^2 + c^2} = \text{const}$$

We have developed expressions for the rate of change of the tangent and binormal vectors. To complete the discussion, we consider the rate of change of the principal normal vector with respect to arc length. Since $\vec{n}$ is a unit vector, $d\vec{n}/ds$ is orthogonal to $\vec{n}$. From (4–17),

$$\vec{b} \cdot \frac{d\vec{n}}{ds} = \tau \qquad \text{(a)}$$

To determine the component of $d\vec{n}/ds$ in the $\vec{t}$ direction, we differentiate the orthogonality relation, $\vec{t} \cdot \vec{n} = 0$.

$$\vec{t} \cdot \frac{d\vec{n}}{ds} = -n \cdot \frac{d\vec{t}}{ds} = -K \qquad \text{(b)}$$

It follows from (a) and (b) that

$$\frac{d\vec{n}}{ds} = -K\vec{t} + \tau\vec{b} \qquad (4\text{–}18)$$

The differentiation formulas for $\vec{t}$, $\vec{n}$, and $\vec{b}$ are called the Frenet equations.

4–6. SUMMARY OF THE GEOMETRICAL RELATIONS FOR A SPACE CURVE

We summarize the geometrical relations for a space curve:

Orthogonal Unit Vectors

$$\begin{aligned}
\vec{t} &= \frac{d\vec{r}}{ds} = \frac{1}{\alpha}\frac{d\vec{r}}{dy} = \text{tangent vector} \\
\vec{n} &= \frac{1}{\left|\frac{d\vec{t}}{dy}\right|}\frac{d\vec{t}}{dy} = \text{principal normal vector} \\
\vec{b} &= \vec{t} \times \vec{n} = \text{binormal vector} \\
\alpha &= \left|\frac{d\vec{r}}{dy}\right| = \frac{ds}{dy}
\end{aligned} \qquad (4\text{–}19)$$

Differentiation Formulas (Frenet Equations)

$$\begin{aligned}
\frac{d\vec{t}}{ds} &= \frac{1}{\alpha}\frac{d\vec{t}}{dy} = K\vec{n} \\
\frac{d\vec{b}}{ds} &= \frac{1}{\alpha}\frac{d\vec{b}}{dy} = -\tau\vec{n} \\
\frac{d\vec{n}}{ds} &= \frac{1}{\alpha}\frac{d\vec{n}}{dy} = -K\vec{t} + \tau\vec{b} \\
K &= \frac{1}{\alpha}\left|\frac{d\vec{t}}{dy}\right| = \text{curvature} \\
\tau &= \frac{1}{\alpha}\vec{b} \cdot \frac{d\vec{n}}{dy} = \text{torsion}
\end{aligned} \qquad (4\text{–}20)$$

We use the orthogonal unit vectors ($\vec{t}$, $\vec{n}$, $\vec{b}$) to define the local reference frame for a member element. This is discussed in the following sections. The Frenet equations are utilized to establish the governing differential equations for a member element.

4–7. LOCAL REFERENCE FRAME FOR A MEMBER ELEMENT

The reference frame associated with $\vec{t}$, $\vec{n}$, and $\vec{b}$ at a point, say P, on a curve is uniquely defined once the curve is specified, that is, it is a property of the curve. We refer to this frame as the *natural* frame at P. The components of the unit vectors $(\vec{t}, \vec{n}, \vec{b})$ are actually the direction cosines for the natural frame with respect to the basic cartesian frame which is defined by the orthogonal unit vectors $(\vec{i}_1, \vec{i}_2, \vec{i}_3)$. We write the relations between the unit vectors as

$$\begin{Bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{Bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{Bmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{Bmatrix} \qquad (4\text{–}21)$$

One can express* the direction cosines in terms of derivatives of the cartesian coordinates (x_1, x_2, x_3) by expanding (4–19). Since $(\vec{t}, \vec{n}, \vec{b})$ are mutually orthogonal unit vectors (as well as $\vec{i}_1, \vec{i}_2, \vec{i}_3$) the direction cosines are related by

$$\sum_{m=1}^{3} \ell_{jm}\ell_{km} = \delta_{jk} \qquad j, k = 1, 2, 3 \qquad (4\text{–}22)$$

Equation (4–22) leads to the important result

$$[\ell_{jk}]^T = [\ell_{jk}]^{-1} \qquad (4\text{–}23)$$

and we see that $[\ell_{jk}]$ is an *orthogonal* matrix.†

The results presented above are applicable to an arbitrary continuous curve. Now, we consider the curve to be the reference axis for a member element and take the positive tangent direction and two orthogonal directions in the normal plane as the directions for the *local* member frame. We denote the directions of the local frame by (Y_1, Y_2, Y_3) and the corresponding unit vectors by $(\vec{t}_1, \vec{t}_2, \vec{t}_3)$. We will always take the positive tangent direction as the Y_1 direction $(\vec{t}_1 = \vec{t})$ and we work only with right handed systems $(\vec{t}_1 \times \vec{t}_2 = \vec{t}_3)$. This notation is shown in Fig. 4–6.

When the centroid of the normal cross-section coincides with the origin of the local frame (point P in Fig. 4–6) at every point, the reference axis is called the centroidal axis for the member. It is convenient, in this case, to take Y_2, Y_3 as the principal inertia directions for the cross section.

In general, we can specify the orientation of the local frame with respect to the natural frame in terms of the angle ϕ between the principal normal direction and the Y_2 direction. The unit vectors defining the local and natural frames

* See Prob. 4–5.
† See Prob. 4–6.

are related by

$$\begin{aligned} \vec{t}_1 &= \vec{t} \\ \vec{t}_2 &= \cos\phi\vec{n} + \sin\phi\vec{b} \\ \vec{t}_3 &= -\sin\phi\vec{n} + \cos\phi\vec{b} \end{aligned} \qquad (4\text{–}24)$$

Combining (4–21) and (4–24) and denoting the product of the two direction cosine matrices by $\boldsymbol{\beta}$, the relation between the unit vectors for the local and basic frames takes the concise form

$$\mathbf{t} = \boldsymbol{\beta}\mathbf{i} \qquad (4\text{–}25)$$

where

$$\boldsymbol{\beta} = \left[\begin{array}{c|c|c} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21}\cos\phi + \ell_{31}\sin\phi & \ell_{22}\cos\phi + \ell_{32}\sin\phi & \ell_{23}\cos\phi + \ell_{33}\sin\phi \\ -\ell_{21}\sin\phi + \ell_{31}\cos\phi & -\ell_{22}\sin\phi + \ell_{32}\cos\phi & -\ell_{23}\sin\phi + \ell_{33}\cos\phi \end{array}\right]$$

Note that the elements of $\boldsymbol{\beta}$ are the direction cosines for the local frame with respect to the basic frame.

$$\beta_{jk} = \vec{t}_j \cdot \vec{i}_k = \cos(Y_j, X_k) \qquad (4\text{–}26)$$

Since both frames are orthogonal, $\boldsymbol{\beta}^T = \boldsymbol{\beta}^{-1}$. We will utilize (4–25) in the next chapter to establish the transformation law for the components of a vector.

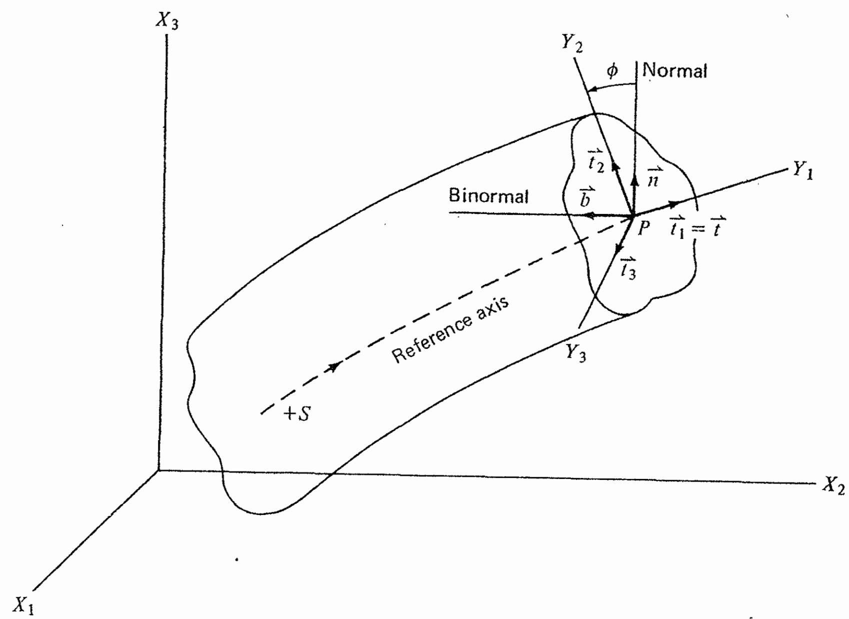


Fig. 4–6. Definition of local reference frame for the normal cross section.

Example 4–6

We determine $\boldsymbol{\beta}$ for the circular helix. The natural frame is related to the basic frame by

$$\begin{Bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{Bmatrix} = \begin{bmatrix} -\frac{a}{\alpha}\sin y & \frac{a}{\alpha}\cos y & \frac{c}{\alpha} \\ -\cos y & -\sin y & 0 \\ \frac{c}{\alpha}\sin y & -\frac{c}{\alpha}\cos y & \frac{a}{\alpha} \end{bmatrix} \begin{Bmatrix} \vec{i}_1 \\ \vec{i}_2 \\ \vec{i}_3 \end{Bmatrix} = [\ell_{jk}]\{\vec{i}_k\}$$

Using (4–25)

$$\boldsymbol{\beta} = \left[\begin{array}{c|c|c} -\frac{a}{\alpha}\sin y & \frac{a}{\alpha}\cos y & \frac{c}{\alpha} \\ -\cos y\cos\phi + \frac{c}{\alpha}\sin y\sin\phi & -\sin y\cos\phi - \frac{c}{\alpha}\cos y\sin\phi & \frac{a}{\alpha}\sin\phi \\ +\cos y\sin\phi + \frac{c}{\alpha}\sin y\cos\phi & \sin y\sin\phi - \frac{c}{\alpha}\cos y\cos\phi & \frac{a}{\alpha}\cos\phi \end{array}\right]$$

4–8. CURVILINEAR COORDINATES FOR A MEMBER ELEMENT

We take as curvilinear coordinates (y_1, y_2, y_3) for a point, say Q, the parameter y_1 of the reference axis and the coordinates (y_2, y_3) of Q with respect to the orthogonal directions (Y_2, Y_3) in the normal cross section (see Fig. 4–7). Let $\vec{R} = \vec{R}(y_1, y_2, y_3)$ be the position vector for $Q(y_1, y_2, y_3)$ and $\vec{r} = \vec{r}(y_1)$ the position vector for the reference axis. They are related by

$$\vec{R} = \vec{r} + y_2\vec{t}_2 + y_3\vec{t}_3$$

where

$$\begin{aligned} \vec{t}_2 &= \vec{t}_2(y_1) = \cos\phi\vec{n} + \sin\phi\vec{b} \\ \vec{t}_3 &= \vec{t}_3(y_1) = -\sin\phi\vec{n} + \cos\phi\vec{b} \end{aligned} \tag{4–27}$$

We consider ϕ to be a function of y_1.

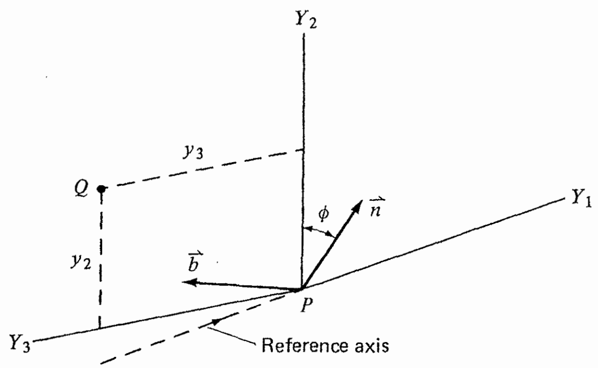


Fig. 4–7. Curvilinear coordinates for the cross section.

The curve through point Q corresponding to increasing y_1 with y_2 and y_3 held constant is called the parametric curve (or line) for y_1. In general, there are three parametric curves through a point. We define $\vec{u}_j$ as the unit tangent vector for the y_j parametric curve through Q. By definition,

$$\begin{aligned} \vec{u}_j &= \frac{1}{g_j}\frac{\partial\vec{R}}{\partial y_j} \\ g_j &= \left|\frac{\partial\vec{R}}{\partial y_j}\right| \end{aligned} \tag{4–28}$$

The differential arc length along the y_j curve is related to dy_j by

$$ds_j = \left|\frac{\partial\vec{R}}{\partial y_j}\right| dy_j = g_j\, dy_j \tag{4–29}$$

This notation is illustrated in Fig. 4–8. One can consider the vectors $\vec{u}_j$ (or $\partial\vec{R}/\partial y_j$) to define a local reference frame at Q.

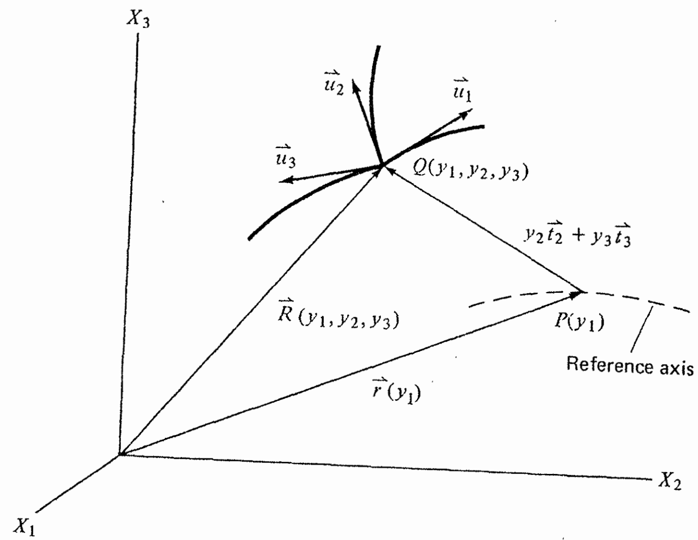


Fig. 4–8. Vectors defining the curvilinear directions.

Operating on (4–27), the partial derivatives of $\vec{R}$ are

$$\begin{aligned} \frac{\partial\vec{R}}{\partial y_1} &= \frac{d\vec{r}}{dy_1} + y_2\frac{d\vec{t}_2}{dy_1} + y_3\frac{d\vec{t}_3}{dy_1} \\ \frac{\partial\vec{R}}{\partial y_2} &= \vec{t}_2 \\ \frac{\partial\vec{R}}{\partial y_3} &= \vec{t}_3 \end{aligned} \tag{a}$$

We see that

$$\vec{u}_2 = \vec{t}_2 \qquad g_2 = 1$$
$$\vec{u}_3 = \vec{t}_3 \qquad g_3 = 1 \tag{4-30}$$

It remains to determine $\vec{u}_1$ and g_1.

Now,

$$\frac{d\vec{r}}{dy_1} = \alpha\vec{t} = \alpha\vec{t}_1 \tag{a}$$

Also,

$$\frac{d\vec{t}_2}{dy_1} = \cos\phi\left(\frac{d\vec{n}}{dy_1} + \frac{d\phi}{dy_1}\vec{b}\right) + \sin\phi\left(\frac{d\vec{b}}{dy_1} - \vec{n}\frac{d\phi}{dy_1}\right)$$
$$\frac{d\vec{t}_3}{dy_1} = -\sin\phi\left(\frac{d\vec{n}}{dy_1} + \vec{b}\frac{d\phi}{dy_1}\right) + \cos\phi\left(\frac{d\vec{b}}{dy_1} - \vec{n}\frac{d\phi}{dy_1}\right) \tag{b}$$

We use the Frenet equations to expand the derivatives of $\vec{n}$ and $\vec{b}$. Then,

$$\frac{d\vec{t}_2}{dy_1} = -\alpha K\cos\phi\,\vec{t}_1 + \left(\alpha\tau + \frac{d\phi}{dy}\right)\vec{t}_3$$
$$\frac{d\vec{t}_3}{dy_1} = \alpha K\sin\phi\,\vec{t}_1 - \left(\alpha\tau + \frac{d\phi}{dy}\right)\vec{t}_2 \tag{c}$$

and finally,

$$\frac{\partial\vec{R}}{\partial y_1} = \alpha(1 - Ky_2')\vec{t}_1 + \left(\alpha\tau + \frac{d\phi}{dy}\right)(y_2\vec{t}_3 - y_3\vec{t}_2) \tag{4-31}$$
$$y_2' = y_2\cos\phi - y_3\sin\phi$$

We see from Fig. 4-9 that y_2' is the coordinate of the point with respect to the principal normal direction.

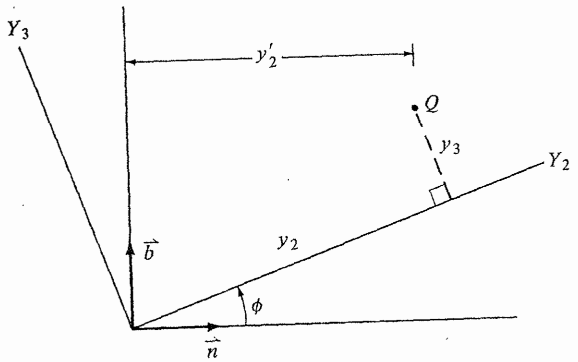


Fig. 4-9. Definition of y_2'.

Since $\partial\vec{R}/\partial y_1$ (and therefore $\vec{u}_1$) involve $\vec{t}_2$ and $\vec{t}_3$, the reference frame defined by $\vec{u}_1, \vec{u}_2, \vec{u}_3$ will not be orthogonal. However, we can reduce it to an orthogonal

system by taking

$$\frac{d\phi}{dy} = -\alpha\tau \tag{4-32}$$

which requires

$$\phi = -\int_{y_0}^{y} \alpha\tau\, dy \tag{4-33}$$

When (4-32) is satisfied,

$$\frac{\partial\vec{R}}{\partial y_1} = \alpha(1 - Ky_2')\vec{t}_1$$

and

$$\vec{u}_1 = \vec{t}_1$$
$$g_1 = \alpha(1 - Ky_2') \tag{4-34}$$

In this case, the local frame at Q coincides with the frame at the centroid. One should note that this simplification is practical only when $\alpha\tau$ can be readily integrated.

Example 4-7

The parameters α and τ are constant for a circular helix:

$$\alpha = (a^2 + c^2)^{1/2}$$
$$\tau = \frac{c}{\alpha^2}$$

Then,

$$\alpha\tau = \frac{c}{\alpha}$$

and integrating (4-33), we obtain

$$\phi = -\frac{c}{\alpha}(y - y_0) = -\tau s$$

For this curve, ϕ varies linearly with y (or arc length). The parameter g_1 follows from (4-34).

$$g_1 = \frac{ds_1}{dy_1} = \alpha(1 - Ky_2')$$
$$= \alpha\left(1 - \frac{a}{\alpha^2}y_2'\right)$$

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PROBLEMS

4–1. Determine $\vec{t}, \vec{n}, \vec{b}, \alpha, K, \tau$ for the following curves:

(a) $x_1 = 3\cos y \quad x_2 = 3\sin y \quad x_3 = 5y$

(b) $x_1 = 3\cos y \quad x_2 = 6\sin y \quad x_3 = 5y$

(c) $\vec{r} = y\vec{i}_1 + y^2\vec{i}_2 + y^3\vec{i}_3$

(d) $x_1 = ae^{\beta y}\cos y$
$x_2 = ae^{\beta y}\sin y$
$x_3 = cy$
where a, β, c are real constants.

4–2. If $x_3 \equiv 0$, the curve lies in the X_1-X_2 plane. Then, $\tau \equiv 0$ and $b = \pm i_3$. The sign of $\vec{b}$ will depend on the relative orientation of $\vec{n}$ with respect to $\vec{t}$. Suppose the equation defining the curve is expressed in the form

$$x_2 = f(x_1) \qquad x_3 = 0 \tag{a}$$

Equation (a) corresponds to taking x_1 as the parameter for the curve.

(a) Determine the expressions for $\vec{t}, \vec{n}, \vec{b}, \alpha$, and K corresponding to this representation. Note that $x_1 \equiv y$ and $\vec{r} = x_1\vec{i}_1 + f(x_1)\vec{i}_2 + 0\vec{i}_3$. Let

$$\frac{df}{dx_1} = f' \qquad \frac{d^2f}{dx_1^2} = f'' \text{ etc.}$$

(b) Apply the results of (a) to

$$x_2 = \frac{4a}{b^2}(x_1 b - x_1^2)$$

where a and b are constants. This is the equation for a parabola symmetrical about $x_1 = b/2$.

(c) Let θ be the angle between $\vec{t}$ and $\vec{i}_1$.

$$\cos\theta = \vec{t}\cdot\vec{i}_1.$$

Deduce that $\alpha = \sec\theta$. Express $\vec{t}, \vec{n}, \vec{b}$, and K in terms of θ.

(d) Specialize (c) for the case where θ^2 is negligible with respect to unity. This approximation leads to

$$\sin\theta \approx \tan\theta \approx \theta$$
$$\cos\theta \approx 1$$

A curve is said to be shallow when $\theta^2 \ll 1$.

4–3. Let $K = 1/R$ and $\tau = 1/R_t$. Show that (see (4–20))

$$\frac{d\vec{t}}{dy} = \frac{\alpha}{R}\vec{n}$$
$$\frac{d\vec{b}}{dy} = -\frac{\alpha}{R_t}\vec{n}$$
$$\frac{d\vec{n}}{dy} = -\frac{\alpha}{R}\vec{t} + \frac{\alpha}{R_t}\vec{b}$$

4–4. The equations for an ellipse can be written as

$$x_1 = a\cos y \qquad x_2 = b\sin y \tag{a}$$

or

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \tag{b}$$

Determine $\alpha, \vec{t}, \vec{n}$ for both parametric representations. Take x_1 as the parameter for (b). Does y have any geometrical significance?

4–5. Show that

$$\ell_{1k} = \frac{1}{\alpha}\frac{dx_k}{dy} = \frac{dx_k}{ds}$$

$$\ell_{2k} = \frac{\dfrac{d\ell_{1k}}{dy}}{\left[\sum_{k=1}^{3}\left(\dfrac{d\ell_{1k}}{dy}\right)^2\right]^{1/2}} = \frac{\dfrac{d^2x_k}{dy^2} - \dfrac{1}{\alpha}\dfrac{d\alpha}{dy}\dfrac{dx_k}{dy}}{\left[\sum_{k=1}^{3}\left(\dfrac{d^2x_k}{dy^2} - \dfrac{1}{\alpha}\dfrac{d\alpha}{dy}\dfrac{dx_k}{dy}\right)^2\right]^{1/2}}$$

$$\begin{Bmatrix}\ell_{31}\\ \ell_{32}\\ \ell_{33}\end{Bmatrix} = \begin{bmatrix}0 & -\ell_{13} & \ell_{12}\\ \ell_{13} & 0 & -\ell_{11}\\ -\ell_{12} & \ell_{11} & 0\end{bmatrix}\begin{Bmatrix}\ell_{21}\\ \ell_{22}\\ \ell_{23}\end{Bmatrix}$$

4–6. Let

$$\ell_j = [\ell_{j1} \quad \ell_{j2} \quad \ell_{j3}]$$

Then,

$$[\ell_{jk}] = \begin{bmatrix}\ell_1\\ \ell_2\\ \ell_3\end{bmatrix}$$

Using (4–22), show that

$$[\ell_{jk}]^T = [\ell_{jk}]^{-1}$$

4–7. Determine $\boldsymbol{\beta}$ for Prob. 4–1a.

4–8. Determine $\boldsymbol{\beta}$ for Prob. 4–1b.

4–9. Specialize $\boldsymbol{\beta}$ for the case where the reference axis is in the $X_1 - X_2$ plane. Note that $\vec{b} = \ell_{33}\vec{i}_3$ where $|\ell_{33}| = 1$. When the reference axis is a plane curve *and* $\phi = 0$, we call the member a "planar" member.

4–10. We express the differentiation formulas for $\vec{i}_j$ *as*

$$\frac{d\mathbf{t}}{ds} = \mathbf{at}$$

(a) Show that $\mathbf{a}$ is, in general, skewsymmetric for an orthogonal system of unit vectors, i.e., $\vec{i}_j \cdot \vec{i}_k = \delta_{jk}$. Determine $\mathbf{a}$.

(b) Suppose the reference axis is a plane curve but $\phi \neq 0$. The member is *not* planar in this case. Determine $\mathbf{a}$.