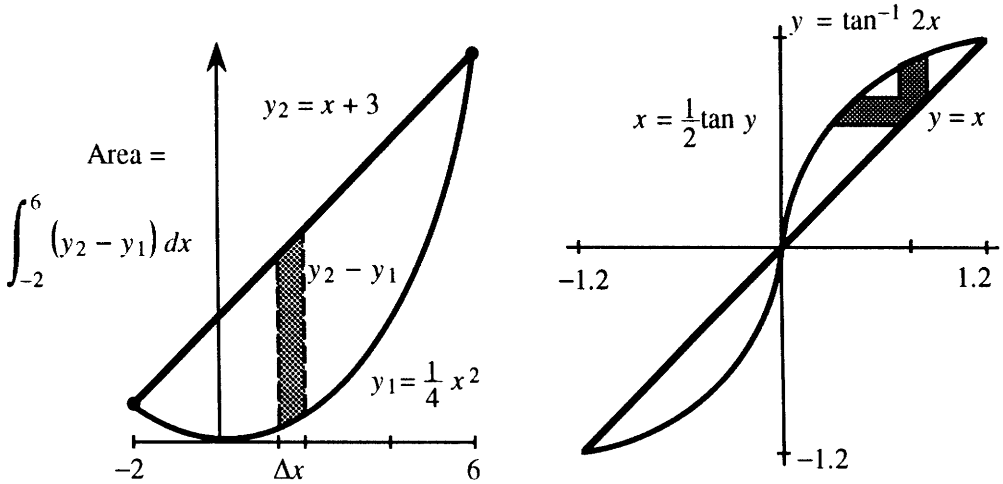
CHAPTER 8 APPLICATIONS OF THE INTEGRAL

8.1 Areas and Volumes by Slices (page 318)

1. Find the area of the region enclosed by the curves $y_1 = \frac{1}{4}x^2$ and $y_2 = x + 3$.

   - The first step is to sketch the region. Find the points where the curves intersect. In this case $\frac{1}{4}x^2 = x + 3$ gives $x = 6$ and $x = -2$. (You need these for your limits of integration.) A thin rectangle between the curves has area $(y_2 - y_1)\Delta x$. Because the rectangle is vertical, its thin side is Δx (not Δy). Then you integrate with respect to x. The area integral is $\int_{-2}^{6}(x + 3 - \frac{1}{4}x^2)dx = 10\frac{2}{3}$.



2. Find the area between the graphs of $y = \tan^{-1}(2x)$ and $y = x$.

   - A graphing calculator shows that these curves intersect near (1.2,1.2) and $(-1.2, -1.2)$. The plan is to find the area in the first quadrant and double it. This shortcut is possible because of the symmetry in the graph. We will use horizontal strips.

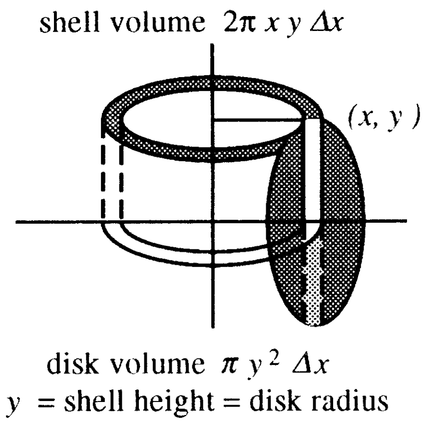
     The length of the horizontal strip is $x_2 - x_1$, where x_2 is on the straight line and x_1 is on the arctan graph. Therefore $x_2 = y$ and $x_1 = \frac{1}{2}\tan y$. Since the thin part of the rectangle is Δy, the area is an integral with respect to y:

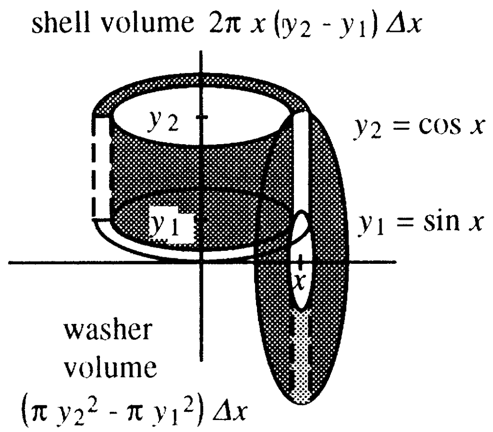
$$\text{Area } = 2 \cdot \int_0^{1.2} (y - \frac{1}{2}\tan y)\,dy = 2(\frac{1}{2}y^2 + \frac{1}{2}\ln\cos y)]_0^{1.2} \approx 0.42.$$

Area and volume can be found using vertical or horizontal strips. It is good to be familiar with both, because sometimes one method is markedly easier than the other. With vertical strips, Problem 2 would integrate $\tan^{-1}(2x)$. With horizontal strips it was $\frac{1}{2}\tan y$. Those functions are *inverses* (which we expect in switching between x and y).

The techniques for area are also employed for volume – especially for solids of revolution. A sketch is needed to determine the intersection points of the curves. Draw a thin strip – horizontal (Δy) or vertical (Δx) – whichever seems easier. Imagine the path of the strip as it is revolved. If the strip is perpendicular to the axis of revolution, the result is a thin disk or a washer. For a disk use $V = \int \pi y^2 dx$ or $V = \int \pi x^2 dy$. *A washer is a difference of circles.* Integrate $\pi(y_2^2 - y_1^2)$ or $\pi(x_2^2 - x_1^2)$.

If the strip is parallel to the axis, rotation produces a cylindrical shell. The volume of the shell is $\int 2\pi$ (radius) (height) (thickness). For an upright shell this is $\int 2\pi x\ y\ dx$.





3. (This is 8.1.24) Find the volume when the first-quadrant region bounded by $y = \sin x, y = \cos x$ and $x = 0$ is revolved around (a) the x axis and (b) the y axis.

   - (a) $y_1 = \sin x$ intersects $y_2 = \cos x$ at $x = \frac{\pi}{4}, y = \frac{\sqrt{2}}{2}$. A vertical strip is sketched. Imagine it revolving about the x axis. The result is a thin washer whose volume is $(\pi y_2^2 - \pi y_1^2)\Delta x$. This leads to the integral $\pi \int_0^{\pi/4}(\cos^2 x - \sin^2 x)dx$. Since $\cos^2 x - \sin^2 x = \cos 2x$, the total volume is $\pi \int_0^{\pi/4} \cos 2x\, dx = \frac{\pi}{2} \sin 2x]_0^{\pi/4} = \frac{\pi}{2}$.
   - (b) When the same region is revolved about the y axis, the same strip carves out a thin shell. Its volume is approximately **circumference** $\times$ **height** $\times$ $\boldsymbol{\Delta x}$ $= 2\pi r(y_2 - y_1)\Delta x$. The radius of the shell is the distance $r = x$ from the y axis to the strip. The total volume is $V = \int_0^{\pi/4} 2\pi x(\cos x - \sin x)dx$. After integration by parts this is

$$[2\pi x(\sin x + \cos x)]_0^{\pi/4} - \int_0^{\pi/4} 2\pi(\sin x + \cos x)dx = 2\pi(\frac{\pi}{2}\sqrt{2} - 1) \approx 0.70.$$

4. Find the volume when the triangle with vertices (1,1), (4,1), and (6,6) is revolved around (a) the x axis and (b) the y axis.

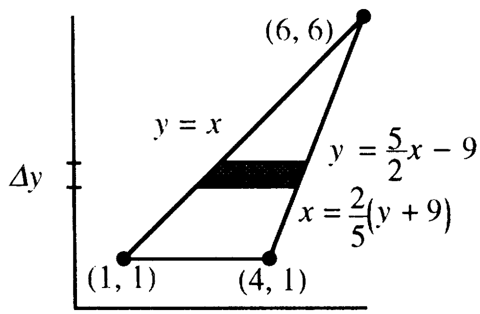
   - A horizontal strip has been drawn between two sides of the triangle. The left side is the line $y = x$, so $x_1 = y$. The right side is the line $y = \frac{5}{2}x - 9$, so $x_2 = \frac{2}{5}(y + 9)$. Since the thin side of the strip is Δy, integrals will be with respect to y.

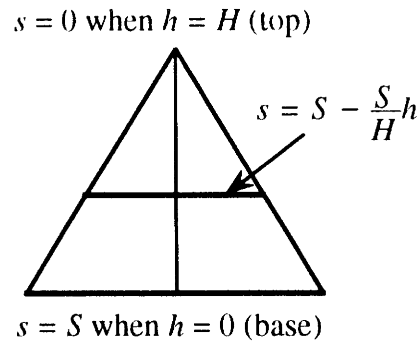
     (a) When the triangle is revolved around the x axis, the strip is parallel to the axis of revolution. This produces a shell of radius y :

$$V = \int 2\pi(\text{radius})\ (\text{height})\ dy = 2\pi \int_1^6 y(\frac{2}{5}(y+9) - y)dy = 40\pi.$$

   - (b) When the region is revolved around the y axis, the path of the strip is a washer (disk with hole). The volume is

$$\int_1^6 \pi(x_2^2 - x_1^2)dy = \pi \int_1^6 ((\frac{2}{5}(y+9))^2 - y^2)dy = 55\pi.$$





5. Find the volume of a pyramid, when the height is H and the square base has sides of length S. This is not a solid of revolution!

- Imagine the pyramid sliced horizontally. Each slice is like a thin square card. The thickness of each card is Δh. The sides of the square decrease in length from $s = S$ when $h = 0$ to $s = 0$ when $h = H$. *We can write* $s = S - \frac{S}{H}h$. That is probably the hardest step.

  The volume of each card is $s^2 \Delta h$ or $(S - \frac{S}{H}h)^2 \Delta h$. If we add those volumes and let $\Delta h \to 0$ (the number of cards goes to ∞) the sum becomes the integral

$$V = \int_0^H (S - \frac{S}{H}h)^2 dh = [-\frac{H}{3S}(S - \frac{S}{H}h)^3]_0^H = \frac{1}{3}S^2 H.$$

Read-throughs and selected even-numbered solutions :

The area between $y = x^3$ and $y = x^4$ equals the integral of $\mathbf{x^3 - x^4}$. If the region ends where the curves intersect, we find the limits on x by solving $\mathbf{x^3 = x^4}$. Then the area equals $\int_0^1 (\mathbf{x^3 - x^4})\mathbf{dx} = \mathbf{\frac{1}{4} - \frac{1}{5} = \frac{1}{20}}$. When the area between $y = \sqrt{x}$ and the y axis is sliced horizontally, the integral to compute is $\int \mathbf{y^2 dy}$.

In three dimensions the volume of a slice is its thickness dx times its **area.** If the cross-sections are squares of side $1 - x$, the volume comes from $\int \mathbf{(1 - x)^2 dx}$. From $x = 0$ to $x = 1$, this gives the volume $\mathbf{\frac{1}{3}}$ of a square **pyramid.** If the cross-sections are circles of radius $1 - x$, the volume comes from $\int \pi\mathbf{(1 - x)^2 dx}$. This gives the volume $\mathbf{\frac{\pi}{3}}$ of a circular **cone.**

For a solid of revolution, the cross-sections are **circles.** Rotating the graph of $y = f(x)$ around the x axis gives a solid volume $\int \pi\mathbf{(f(x))^2 dx}$. Rotating around the y axis leads to $\int \pi\mathbf{(f^{-1}(y))^2 dy}$. Rotating the area between $y = f(x)$ and $y = g(x)$ around the x axis, the slices look like **washers.** Their areas are $\pi\mathbf{(f(x))^2} - \pi\mathbf{(g(x))^2} = \mathbf{A(x)}$ so the volume is $\int \mathbf{A(x)dx}$.

Another method is to cut the solid into thin cylindrical **shells.** Revolving the area under $y = f(x)$ around the y axis, a shell has height $\mathbf{f(x)}$ and thickness dx and volume $\mathbf{2\pi x\, f(x)dx}$. The total volume is $\int \mathbf{2\pi x\, f(x)dx}$.

6 $y = x^{1/5}$ and $y = x^4$ intersect at (0,0) and (1,1): area $= \int_0^1 (x^{1/5} - x^4)dx = \frac{5}{6} - \frac{1}{5} = \mathbf{\frac{19}{30}}$.

12 The region is a curved triangle between $x = -1$ (where $e^{-x} = e$) and $x = 1$ (where $e^x = e$). Vertical strips end at e^{-x} for $x < 0$ and at e^x for $x > 0$: Area $= \int_{-1}^0 (e - e^{-x})dx + \int_0^1 (e - e^x)dx = \mathbf{2}$.

18 Volume $= \int_0^\pi \pi \sin^2 x dx = [\pi(\frac{x - \sin x \cos x}{2})]_0^\pi = \mathbf{\frac{\pi^2}{2}}$.

20 Shells around the y axis have radius x and height $2\sin x$ and volume $(2\pi x)2\sin x dx$. Integrate for the volume of the galaxy: $\int_0^\pi 4\pi x \sin x dx = [4\pi(\sin x - x\cos x)]_0^\pi = \mathbf{8\pi^2}$.

26 The region is a curved triangle, with base between $x = 3, y = 0$ and $x = 9, y = 0$. The top point is where $y = \sqrt{x^2 - 9}$ meets $y = 9 - x$; then $x^2 - 9 = (9 - x)^2$ leads to $x = 5, y = 4$. (a) Around the x axis: Volume $= \int_3^5 \pi(x^2 - 9)dx + \int_5^9 \pi(9 - x)^2 dx = \mathbf{36\pi}$. (b) Around the y axis: Volume $= \int_3^5 2\pi x\sqrt{x^2 - 9}dx + \int_5^9 2\pi x(9 - x)dx = [\frac{2\pi}{3}(x^2 - 9)^{3/2}]_3^5 + [9\pi x^2 - \frac{2\pi x^3}{3}]_5^9 = \frac{2\pi}{3}(64) + 9\pi(9^2 - 5^2) - \frac{2\pi}{3}(9^3 - 5^3) = \mathbf{144\pi}$.

28 The region is a circle of radius 1 with center (2,1). (a) Rotation around the x axis gives a torus with no hole: it is Example 10 with $a = b = 1$ and volume $\mathbf{2\pi^2}$. The integral is $\pi \int_1^3 [(1 + \sqrt{1 - (x - 2)^2}) - (1 - \sqrt{1 - (x - 2)^2}]dx = 4\pi \int_1^3 \sqrt{1 - (x - 2)^2}dx = 4\pi \int_{-1}^1 \sqrt{1 - x^2}dx = 2\pi^2$. (b) Rotation around the y axis also gives a torus. The center now goes around a circle of radius 2 so by Example 10 $V = \mathbf{4\pi^2}$. The volume by shells is $\int_1^3 2\pi x[(1 + \sqrt{1 - (x - 2)^2}) - (1 - \sqrt{1 - (x - 2)^2})]dx = 4\pi \int_1^3 x\sqrt{1 - (x - 2)^2}dx = 4\pi \int_{-1}^1 (x + 2)\sqrt{1 - x^2}dx =$ (odd integral is zero) $8\pi \int_{-1}^1 \sqrt{1 - x^2}dx = 4\pi^2$.

34 The area of a semicircle is $\frac{1}{2}\pi r^2$. Here the diameter goes from the base $y = 0$ to the top edge $y = 1 - x$ of the triangle. So the semicircle radius is $r = \frac{1-x}{2}$. The volume by slices is $\int_0^1 \frac{\pi}{2}(\frac{1-x}{2})^2 dx = [-\frac{\pi}{8}\frac{(1-x)^3}{3}]_0^1 = \mathbf{\frac{\pi}{24}}$.

36 The tilted cylinder has circular slices of area πr^2 (at all heights from 0 to h). So the volume is $\int_0^h \pi r^2 dy = \mathbf{\pi r^2 h}$. This equals the volume of an *untilted* cylinder (Cavalieri's principle: same slice areas give same volume).

40 (a) The slices are **rectangles.** (b) The slice area is $\mathbf{2\sqrt{1 - y^2}}$ **times** $\mathbf{y \tan\theta}$. (c) The volume is $\int_0^1 2\sqrt{1 - y^2}y\tan\theta dy = [-\frac{2}{3}(1 - y^2)^{3/2}\tan\theta]_0^1 = \mathbf{\frac{2}{3}\tan\theta}$. (d) Multiply radius by r and volume by $\mathbf{r^3}$.

50 Volume by shells $= \int_0^2 2\pi x(8 - x^3)dx = [8\pi x^2 - \frac{2\pi}{5}x^5]_0^2 = 32\pi - \frac{64\pi}{5} = \mathbf{\frac{96\pi}{5}}$; volume by horizontal disks $= \int_0^8 \pi(y^{1/3})^2 dy = [\frac{3\pi}{5}y^{5/3}]_0^8 = \frac{3\pi}{5}32 = \mathbf{\frac{96\pi}{5}}$.

56 $\int_1^{100} 2\pi x(\frac{1}{x})dx = 2\pi(99) = \mathbf{198\pi}$.

62 Shells around x axis: volume $= \int_{y=0}^1 2\pi y(1)dy + \int_{y=1}^e 2\pi y(1 - \ln y)dy = [\pi y^2]_0^1 + [\pi y^2 - 2\pi\frac{y^2}{2}\ln y + 2\pi\frac{y^2}{4}]_1^e = \pi + \pi e^2 - \pi e^2 + 2\pi\frac{e^2}{4} - \pi + 0 - 2\pi\frac{1}{4} = \mathbf{\frac{\pi}{2}(e^2 - 1)}$. Check disks: $\int_0^1 \pi(e^x)^2 dx = [\pi\frac{e^{2x}}{2}]_0^1 = \frac{\pi}{2}(e^2 - 1)$.

8.2 Length of a Plane Curve (page 324)

1. (This is 8.2.6) Find the length of $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$.

   - The length is $\int ds = \int_1^2 \sqrt{1 + (\frac{dy}{dx})^2}\, dx$. The square of $\frac{dy}{dx} = x^3 - \frac{1}{4}x^{-3}$ is $(\frac{dy}{dx})^2 = x^6 - \frac{1}{2} + \frac{1}{16}x^{-6}$. Add 1 and take the square root. The arc length is

$$\int_1^2 \sqrt{x^6 + \frac{1}{2} + \frac{1}{16}x^{-6}}\, dx = \int_1^2 (x^3 + \frac{1}{4}x^{-3})dx = \frac{x^4}{4} - \frac{1}{8}x^{-2}]_1^2 = \frac{123}{32}.$$

   In "real life" the expression $\sqrt{1 + (\frac{dy}{dx})^2}$ almost never simplifies as nicely as this. We found a perfect square under the square root sign. Numerical methods are usually required to find lengths of curves.

2. Find the length of $x = \ln\cos y$ from $y = -\pi/4$ to $y = \pi/4$.

   - It seems simpler to find dx/dy than to rewrite the curve as $y = \cos^{-1} e^x$ and compute dy/dx. When you work with dx/dy, use the form $\int ds = \int_{y_1}^{y_2} \sqrt{(\frac{dx}{dy})^2 + 1}\, dy$. In this case $\frac{dx}{dy} = -\tan y$ and $(\frac{dx}{dy})^2 + 1 = \tan^2 y + 1 = \sec^2 y$. Again this has a good square root. The arc length is

$$\int_{-\pi/4}^{\pi/4} \sec y\, dy = \ln(\sec y + \tan y)]_{-\pi/4}^{\pi/4} = \ln(\frac{\sqrt{2}+1}{\sqrt{2}-1}) \approx 1.76.$$

3. The position of a point is given by $x = e^t \cos t, y = e^t \sin t$. Find the length of the path from $t = 0$ to $t = 6$.

- Since the curve is given in parametric form, it is best to use $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$. This example has $dx/dt = -e^t \sin t + e^t \cos t$ and $dy/dt = e^t \cos t + e^t \sin t$. When you square and add, the cross terms involving $\sin t \cos t$ cancel each other. This leaves

$$\frac{ds}{dt} = \sqrt{2e^{2t}\sin^2 t + 2e^{2t}\cos^2 t} = \sqrt{2e^{2t}(\sin^2 t + \cos^2 t)} = \sqrt{2}e^t.$$

The arc length is $\int ds = \int_0^6 \sqrt{2}\, e^t dt = \sqrt{2}(e^6 - 1)$.

Read-throughs and selected even-numbered solutions :

The length of a straight segment (Δx across, Δy up) is $\Delta s = \sqrt{(\mathbf{\Delta x})^2 + (\mathbf{\Delta y})^2}$. Between two points on the graph of $y(x), \Delta y$ is approximately dy/dx times $\mathbf{\Delta x}$. The length of that piece is approximately $\sqrt{(\Delta x)^2 + (\mathbf{dy/dx})^2(\mathbf{\Delta x})^2}$. An infinitesimal piece of the curve has length $ds = \sqrt{\mathbf{1 + (dy/dx)^2}}\, \mathbf{dx}$. Then the arc length integral is $\int \mathbf{ds}$.

For $y = 4 - x$ from $x = 0$ to $x = 3$ the arc length is $\int_{\mathbf{0}}^{\mathbf{3}} \sqrt{\mathbf{2}}\, \mathbf{dx} = \mathbf{3\sqrt{2}}$. For $y = x^3$ the arc length integral is $\int \sqrt{\mathbf{1 + 9x^4}}\, \mathbf{dx}$.

The curve $x = \cos t, y = \sin t$ is the same as $\mathbf{x^2 + y^2 = 1}$. The length of a curve given by $x(t), y(t)$ is $\int \sqrt{(\mathbf{dx/dt})^2 + (\mathbf{dy/dt^2})}dt$. For example $x = \cos t, y = \sin t$ from $t = \pi/3$ to $t = \pi/2$ has length $\int_{\pi/\mathbf{3}}^{\pi/\mathbf{2}} \mathbf{dt}$. The speed is $ds/dt = \mathbf{1}$. For the special case $x = t, y = f(t)$ the length formula goes back to $\int \sqrt{\mathbf{1 + (f'(x))^2}}dx$.

4 $y = \frac{1}{3}(x^2 - 2)^{3/2}$ has $\frac{dy}{dx} = x(x^2 - 2)^{1/2}$ and length $= \int_2^4 \sqrt{1 + x^2(x^2 - 2)}dx = \int_2^4 (x^2 - 1)dx = \mathbf{\frac{50}{3}}$.

10 $\frac{dx}{dt} = \cos t - \sin t$ and $\frac{dy}{dt} = -\sin t - \cos t$ and $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 2$. So length $= \int_0^\pi \sqrt{2}dt = \mathbf{\sqrt{2}}\pi$. The curve is a **half** of a circle of radius $\sqrt{2}$ because $x^2 + y^2 = 2$ and t stops at π.

14 $\frac{dx}{dt} = (1 - \frac{1}{2}\cos 2t)(-\sin t) + \sin 2t \cos t = \frac{3}{2}\sin t \cos 2t$. Note: first rewrite $\sin 2t \cos t = 2\sin t \cos^2 t = \sin t(1 + \cos 2t)$. Similarly $\frac{dy}{dt} = \frac{3}{2}\cos t \cos 2t$. Then $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (\frac{3}{2}\cos 2t)^2$. So length $= \int_0^{\pi/4} \frac{3}{2}\cos 2t dt = \mathbf{\frac{3}{4}}$. This is the only arc length I have ever personally discovered; the problem was meant to have an asterisk.

18 $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 3\cos t$ so length $= \int_0^{2\pi} \sqrt{\sin^2 t + 9\cos^2 t}\, dt =$ perimeter of ellipse. This integral has no closed form. Match it with a table of "elliptic integrals" by writing it as $4\int_0^{\pi/2} \sqrt{9 - 8\sin^2 t}dt = 12\int_0^{\pi/2} \sqrt{1 - \frac{8}{9}\sin^2 t}dt$. The table with $k^2 = \frac{8}{9}$ gives 1.14 for this integral or 12 (1.14) = 13.68 for the perimeter. Numerical integration is the expected route to this answer.

24 The curve $x = y^{3/2}$ is the **mirror image** of $y = x^{3/2}$ in Problem 1: same length $\frac{13^{3/2} - 4^{3/2}}{27}$ (also Problem 2).

28 (a) Length integral $= \int_0^\pi \sqrt{4\cos^2 t \sin^2 t + 4\cos^2 t \sin^2 t}\, dt = \int_0^\pi 2\sqrt{2}|\cos t \sin t|dt = \mathbf{2\sqrt{2}}$. (Notice that $\cos t$ is *negative* beyond $t = \frac{\pi}{2}$: split into $\int_0^{\pi/2} + \int_{\pi/2}^{\pi}$. (b) All points have $x + y = \cos^2 t + \sin^2 t = 1$. (c) The path from (1,0) reaches (0,1) when $t = \frac{\pi}{2}$ and returns to (1,0) at $t = \pi$. Two trips of length $\sqrt{2}$ give $2\sqrt{2}$.

8.3 Area of a Surface of Revolution (page 327)

The area of a surface of revolution is $A = \int 2\pi(\text{radius})\, ds$. Notice ds. The radius is y (or x) if the revolution is around the x (or y) axis. Use the form of ds which is most convenient:

$$ds = \sqrt{1+(\frac{dy}{dx})^2}\,dx \quad \text{or} \quad \sqrt{(\frac{dx}{dy})^2+1}\,dy \quad \text{or} \quad \sqrt{(\frac{dy}{dt})^2+(\frac{dx}{dt})^2}\,dt.$$

Pick the limits of integration which correspond to your choice of dx, dy, or dt.

1. The mirrored surface behind a searchlight bulb can be modelled by revolving the parabola $y = \frac{1}{10}x^2$ around the y axis. Find the area of the surface from $x = 0$ to $x = 20$.

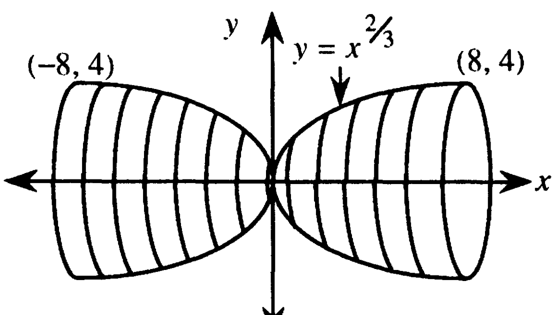
   - The radius around the y axis is x, and $\frac{dy}{dx} = \frac{x}{5}$. The area formula is $A = \int 2\pi x\, ds = 2\pi \int_0^{20} x\sqrt{1+\frac{x^2}{25}}dx$. Integrate by the substitution $u = 1+\frac{x^2}{25}$ with $du = \frac{2}{25}x\,dx$ and $x\,dx = \frac{25}{2}\,du$:

$$A = 2\pi(\frac{25}{2})\int_{u_1}^{u_2} u^{1/2}du = 25\pi(\frac{2}{3}u^{3/2})]_{u_1}^{u_2} = \frac{50}{3}\pi(1+\frac{x^2}{25})^{3/2}]_0^{20} = \frac{50}{3}\pi(17^{3/2}-1) \approx 3618 \text{ square inches.}$$

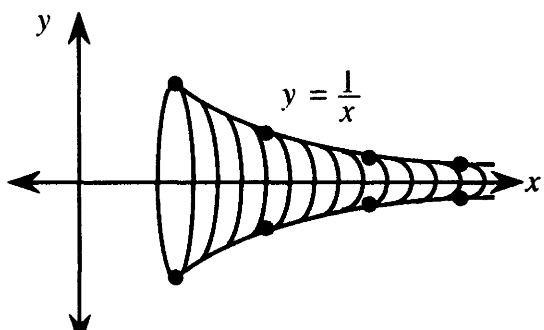
2. The curve $y = x^{2/3}$ from $(-8, 4)$ to $(8,4)$ is revolved around the x axis. Find the surface area of this hourglass.

   - This problem is trickier than it seems because $dy/dx = \frac{2}{3}x^{-1/3}$ is undefined at (0,0). We prefer to avoid that rough point. Since the curve is symmetrical, we can restrict x to $0 \le x \le 8$ and *double the answer.* Then $x = y^{3/2}$, $dx/dy = \frac{3}{2}y^{1/2}$ and area $= 2\int_0^4 2\pi y\sqrt{\frac{9}{4}y+1}\,dy$. Substitute $u = \frac{9}{4}y+1$ and $y = \frac{4}{9}(u-1)$. Then u goes from 1 to 10:

$$A = 4\pi\int_1^{10}\frac{4}{9}(u-1)\sqrt{u}(\frac{4}{9}du) = \frac{64}{81}\pi\int_1^{10}(u^{3/2}-u^{1/2})du = \frac{64}{81}\pi(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2})]_1^{10} \approx 262.3.$$



Problem 2



Problem 3

3. (An infinitely long horn with finite volume but infinite area) Show that when $y = 1/x$, $1 \le x \le \infty$ is revolved around the x axis the surface area is infinite but the volume is finite (Problem 8.3.21).

   - Surface area $= \int_1^\infty 2\pi y\sqrt{1+(dy/dx)^2}dx = \int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx = 2\pi\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}dx.$

   This improper integral diverges by comparison with $\int_1^\infty \frac{1}{x}dx = \ln\infty = \infty$. (Note that $\sqrt{x^4+1} > \sqrt{x^4}$.) Therefore the surface area is infinite.

The disk method gives *volume* $= \pi \int_1^\infty y^2 dx = \pi \int_1^\infty \frac{1}{x^2} dx = \pi(-\frac{1}{x})]_1^\infty = \pi$. Therefore π cubic units of paint would fill this infinite bugle, but all the paint in the world won't cover it.

4. The circle with radius 2 centered at (0,6) is given parametrically by $x = 2\cos t$ and $y = 6 + 2\sin t, 0 \le t \le 2\pi$. Revolving it around the x axis forms a torus (a donut). Find the surface area.

   First a comment. The ordinary xy equation for the circle is $x^2 + (y-6)^2 = 4$ or $y = 6 \pm \sqrt{4 - x^2}$. In this form the circle has two parts. The first part $y = 6 + \sqrt{4 - x^2}$ is the top half of the circle – rotation gives the outer surface of the donut. The inner surface corresponds to $y = 6 - \sqrt{4 - x^2}$. The *parametric form* (using t) takes care of both parts together. As t goes from 0 to 2π, the point $x = 2\cos t, y = 6 + 2\sin t$ goes around the full circle. The surface area is

$$\int_0^{2\pi} 2\pi y\, ds = 2\pi \int_0^{2\pi} (6 + 2\sin t)\sqrt{4\sin^2 t + 4\cos^2 t}\, dt = 2\pi \int_0^{2\pi} (6 + 2\sin t)(2) dt = 48\pi^2.$$

Read-throughs and selected even-numbered solutions :

A surface of revolution comes from revolving a **curve** around **an axis (a line).** This section computes the **surface area.** When the curve is a short straight piece (length Δs), the surface is a **cone.** Its area is $\Delta S = \mathbf{2\pi r \Delta s}$. In that formula (Problem 13) r is the radius of **the circle traveled by the middle point.** The line from (0,0) to (1,1) has length $\mathbf{\Delta s = \sqrt{2}}$, and revolving it produces area $\pi\mathbf{\sqrt{2}}$.

When the curve $y = f(x)$ revolves around the x axis, the area of the surface of revolution is the integral $\int \mathbf{2\pi f(x)\sqrt{1 + (df/dx)^2}dx}$. For $y = x^2$ the integral to compute is $\int \mathbf{2\pi x^2\sqrt{1 + 4x^2}dx}$. When $y = x^2$ is revolved around the y axis, the area is $S = \int \mathbf{2\pi x\sqrt{1 + (df/dx)^2}dx}$. For the curve given by $x = 2t, y = t^2$, change ds to $\mathbf{\sqrt{4 + 4t^2}dt}$.

2 Area $= \int_0^1 2\pi x^3\sqrt{1 + (3x^2)^2}\, dx = [\frac{\pi}{27}(1 + 9x^4)^{3/2}]_0^1 = \mathbf{\frac{\pi}{27}(10^{3/2} - 1)}$

6 Area $= \int_0^1 2\pi \cosh x\sqrt{1 + \sinh^2 x}\, dx = \int_0^1 2\pi\cosh^2 x dx = \int_0^1 \frac{\pi}{2}(e^{2x} + 2 + e^{-2x})dx = [\frac{\pi}{2}(\frac{e^{2x}}{2} + 2x + \frac{e^{-2x}}{-2})]_0^1 =$ $\frac{\pi}{2}(\frac{e^2}{2} + 2 + \frac{e^{-2}}{-2} - 1) = \mathbf{\frac{\pi}{2}(\frac{e^2 - e^{-2}}{2} + 1)}$.

10 Area $= \int_0^1 2\pi x\sqrt{1 + \frac{1}{9}x^{-4/3}}dx$. This is unexpectedly difficult (rotation around the x axis is easier). Substitute $u = 3x^{2/3}$ and $du = 2x^{-1/3}dx$ and $x = (\frac{u}{3})^{3/2}$: Area $= \int_0^3 2\pi(\frac{u}{3})^{3/2}\sqrt{1 + \frac{1}{u^2}}\frac{du}{2}(\frac{u}{3})^{1/2} =$ $\int_0^3 \frac{\pi}{9}u\sqrt{u^2 + 1}du = [\frac{\pi}{27}(u^2 + 1)^{3/2}]_0^3 = \mathbf{\frac{\pi}{27}(10^{3/2} - 1)}$. An equally good substitution is $u = x^{4/3} + \frac{1}{9}$.

14 (a) $dS = 2\pi\sqrt{1 - x^2}\sqrt{1 + \frac{x^2}{1 - x^2}}dx = \mathbf{2\pi dx}$. (b) The area between $x = a$ and $x = a + h$ is $\mathbf{2\pi h}$. All slices of thickness h have this area, whether the slice goes near the center or near the outside. (c) $\frac{1}{4}$ of the Earth's area is above latitude $\mathbf{30°}$ where the height is $R\sin 30° = \frac{R}{2}$. The slice from the Equator up to 30° has the same area (and so do two more slices below the Equator).

20 Area $= \int_{1/2}^1 2\pi x\sqrt{1 + \frac{1}{x^4}}dx = \int_{1/2}^1 2\pi\frac{\sqrt{x^4 + 1}}{x^4}x^3 dx$. Substitute $u = \sqrt{x^4 + 1}$ and $du = 2x^3dx/u$ to find $\int_{\sqrt{17}/4}^{\sqrt{2}} \frac{\pi u^2 du}{u^2 - 1} = [\pi u - \frac{\pi}{2}\ln\frac{u+1}{u-1}]_{\sqrt{17}/4}^{\sqrt{2}} = \pi(\sqrt{2} - \frac{\sqrt{17}}{4} - \frac{1}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{1}{2}\ln\frac{\sqrt{17}+4}{\sqrt{17}-4}) \approx 5.0$.

8.4 Probability and Calculus (page 334)

1. The probability of a textbook page having no errors is $\frac{4}{5}$. The probability of one error is $\frac{4}{25}$ and the probability of n errors is $\frac{4}{5^{n+1}}$. What is the chance that the page will have (a) at least one error (b) fewer than 3 errors (c) more than 5 errors?

   This is a discrete probability problem. The probabilities $\frac{4}{5}, \frac{4}{25}, \frac{4}{125}, \cdots$ add to one.

   - (a) Since the probability of no errors is $\frac{4}{5}$, the probability of 1 or more is $1 - \frac{4}{5} = \frac{1}{5}$.
   - (b) Add the probabilities of zero, one, and two errors: $\frac{4}{5} + \frac{4}{25} + \frac{4}{125} = \frac{124}{125} = 99.2\%$.
   - (c) There are two ways to consider more than five errors. The first is to add the probabilities of 0, 1, 2, 3, 4, and 5 errors. The sum is .999936 and subtraction leaves $1 - .999936 = .000064$. The second way is to add the infinite series $P(6) + P(7) + P(8) + \cdots = \frac{4}{5^7} + \frac{4}{5^8} + \frac{4}{5^9} + \cdots$. Use the formula for a geometric series:

   $$\text{sum} = \frac{a}{1-r} = \frac{\text{first term}}{1-(\text{ratio})} = \frac{4/5^7}{1-\frac{1}{5}} = 0.000064.$$

2. If $p(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$, find the approximate probability that

   (a) $-1 \le x \le 1$ (b) $x \ge 10$ (c) $x \ge 0$.

   - This is continuous probability. Instead of adding p_n, we integrate $p(x)$:

   (a) $\frac{1}{\pi}\int_{-1}^{1} \frac{dx}{1+x^2}$ (b) $\frac{1}{\pi}\int_{10}^{\infty} \frac{dx}{1+x^2}$ (c) $\frac{1}{\pi}\int_{0}^{\infty} \frac{1}{1+x^2}dx$.

   The integral of $\frac{1}{1+x^2}$ is $\arctan x$. Remember $\tan\frac{\pi}{4} = 1$ and $\tan(-\frac{\pi}{4}) = -1$:

   (a) $\frac{1}{\pi}\arctan x]_{-1}^{1} = \frac{1}{2}$ (b) $\frac{1}{\pi}\arctan x]_{10}^{\infty} \approx 0.0317$ (c) $\frac{1}{\pi}\int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}$.

   Part (c) is common sense. Since $\frac{1}{1+x^2}$ is symmetrical (or *even*), the part $x \ge 0$ gives half the area.

3. If $p(x) = \frac{k}{x^4}$ is the probability density for $x \ge 1$, find the constant k.

   - The key is to remember that the total probability must be one:

   $$\int_1^{\infty} \frac{k}{x^4}dx = 1 \text{ yields } [-\frac{k}{3}x^{-3}]_1^{\infty} = \frac{k}{3} = 1. \text{ Therefore } k = 3.$$

4. Find the *mean* (the expected value = average x) for $p(x) = \frac{3}{x^4}$.

   - This is a continuous probability, so use the formula $\mu = \int xp(x)dx$:

   $$\text{The mean is } \mu = \int_1^{\infty} \frac{3x}{x^4}dx = -\frac{3}{2}x^{-2}]_1^{\infty} = \frac{3}{2}.$$

5. Lotsa Pasta Restaurant has 50 tables. On an average night, 5 parties are no-shows. So they accept 55 reservations. What is the probability that they will have to turn away a party with a reservation?

   - This is like Example 3 in the text, which uses the Poisson model. X is the number of no-shows and the average is $\lambda = 5$. The number of no-shows is n with probability $p_n = \frac{5^n}{n!}e^{-5}$. A party is turned away if $n < 5$. Compute $p_0 + p_1 + p_2 + p_3 + p_4 = e^{-5} + 5e^{-5} + \frac{25}{2}e^{-5} + \frac{125}{6}e^{-5} + \frac{625}{24}e^{-5} \approx 0.434$. In actual fact the overflow parties are sent to the bar for a time $t \to \infty$.

6. Lotsa Pasta takes no reservations for lunch. The average waiting time for a table is 20 minutes. What are your chances of being seated in 15 minutes?

   - This is like Example 4. Model the waiting time by $p(x) = \frac{1}{20}e^{-x/20}$ for $0 \le x \le \infty$. The mean waiting time is 20. The chances of being seated in 15 minutes are about 50-50:

$$P(0 \le x \le 15) = \int_0^{15} \frac{1}{20}e^{-x/20}dx = -e^{-x/20}\Big]_0^{15} \approx 0.528.$$

7. A random sample of 1000 baseballs in the factory found 62 to be defective. Estimate with 95% confidence the true percentage of defective baseballs.

   - Since a baseball is either defective or not, this is like a yes-no poll (Example 7). The 95% margin of error is never more than $\frac{1}{\sqrt{N}}$, in this case $\frac{1}{\sqrt{1000}} \approx 0.03$. The "poll" of baseballs found 6.2% defective. We are 95% sure that the true percentage is between $6.2 - .03 = 6.17\%$ and $6.2 + .03 = 6.23\%$.

     Amazingly, the error formula $\frac{1}{\sqrt{N}}$ does not depend on the size of the total population. A poll of 400 people has a margin of error no more than $\frac{1}{\sqrt{400}} = \frac{1}{20} = 5\%$, whether the 400 people are from a city of 100,000 or a country of 250 million. The catch is that the sample must be truly random – very hard to do when dealing with people.

8. Grades on the English placement exam at Absorbine Junior College are normally distributed. The mean is $\mu = 72$ and the standard deviation is $\sigma = 10$. The top 15% are placed in Advanced English. What test grade is the cutoff?

   The distribution of grades looks like Figure 8.12a in the text, except that $\mu = 72$ and $\sigma = 10$. Figure 8.12b gives the area under the curve from ∞ to x. From the graph we see that 84% of the scores are at or below $\mu + \sigma = 82$. Therefore 16% are above, close to the desired 15%. The cutoff is a little over 82. (Statisticians have detailed tables for the area $F(x)$.)

***Read-throughs and selected even-numbered solutions*:**

Discrete probability uses counting, **continuous** probability uses calculus. The function $p(x)$ is the probability **density.** The chance that a random variable falls between a and b is $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{p(x)dx}$. The total probability is $\int_{-\infty}^{\infty} p(x)dx = \mathbf{1}$. In the discrete case $\sum p_n = \mathbf{1}$. The mean (or expected value) is $\mu = \int \mathbf{xp(x)dx}$ in the continuous case and $\mu = \sum np_n$ in the **discrete case.**

The Poisson distribution with mean λ has $p_n = \lambda^{\mathbf{n}}\mathbf{e}^{-\lambda}/\mathbf{n}!$. The sum $\sum p_n = 1$ comes from the **exponential** series. The exponential distribution has $p(x) = e^{-x}$ or $2e^{-2x}$ or $\mathbf{ae^{-ax}}$. The standard Gaussian (or **normal**) distribution has $\sqrt{2\pi}p(x) = e^{-x^2/2}$. Its graph is the well-known **bell-shaped** curve. The chance that the variable falls below x is $F(x) = \int_{-\infty}^{\mathbf{x}} \mathbf{p(x)dx}$. F is the **cumulative** density function. The difference $F(x + dx) - F(x)$ is about $\mathbf{p(x)dx}$, which is the chance that X is between x and $x + dx$.

The *variance,* which measures the spread around μ, is $\sigma^2 = \int \mathbf{(x - \mu)^2 p(x)dx}$ in the continuous case and $\sigma^2 = \sum \mathbf{(n - \mu)^2 p_n}$ in the discrete case. Its square root σ is the **standard deviation.** The normal distribution has $p(x) = \mathbf{e^{-(x-\mu)^2/2\sigma^2}/\sqrt{2\pi}\sigma}$. If $\overline{X}$ is the **average** of N samples from any population with mean μ and variance σ^2, the Law of Averages says that $\overline{X}$ will approach **the mean** μ. The Central Limit Theorem says that the distribution for $\overline{X}$ approaches **a normal distribution.** Its mean is μ and its variance is $\sigma^2/\mathbf{N}$.

In a yes-no poll when the voters are 50-50, the mean for one voter is $\mu = 0(\frac{1}{2}) + 1(\frac{1}{2}) = \mathbf{\frac{1}{2}}$. The variance is $(0-\mu)^2p_0 + (1-\mu)^2p_1 = \mathbf{\frac{1}{4}}$. For a poll with $N = 100, \bar{\sigma}$ is $\sigma/\sqrt{\mathbf{N}} = \mathbf{\frac{1}{20}}$. There is a 95% chance that $\overline{X}$ (the fraction saying yes) will be between $\mu - 2\bar{\sigma} = \mathbf{\frac{1}{2}} - \mathbf{\frac{1}{10}}$ and $\mu + 2\bar{\sigma} = \mathbf{\frac{1}{2}} + \mathbf{\frac{1}{10}}$.

2 The probability of an odd $X = 1, 3, 5, \cdots$ is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \mathbf{\frac{2}{3}}$. The probabilities $p_n = (\frac{1}{3})^n$ **do not add to 1.** They add to $\frac{1}{3} + \frac{1}{9} + \cdots = \frac{1}{2}$ so the adjusted $p_n = 2(\frac{1}{3})^n$ add to 1.

12 $\mu = \int_0^\infty xe^{-x}dx = uv - \int v\,du = -xe^{-x}]_0^\infty + \int_0^\infty e^{-x}dx = \mathbf{1}$.

20 (a) Heads and tails are still equally likely. (b) The coin is still fair so the expected fraction of heads during the second N tosses is $\frac{1}{2}$ and the expected fraction overall is $\mathbf{\frac{1}{2}}(\alpha + \mathbf{\frac{1}{2}})$; which is the average.

28 $\mu = (p_1 + p_2 + p_3 + \cdots) + (p_2 + p_3 + p_4 + \cdots) + (p_3 + p_4 + \cdots) + \cdots = (1) + (\frac{1}{2}) + (\frac{1}{4}) + \cdots = \mathbf{2}$.

32 $2000 \pm 2\sigma$ gives **1700 to 2300** as the 95% confidence interval.

34 The average has mean $\bar{\mu} = 30$ and deviation $\bar{\sigma} = \frac{8}{\sqrt{N}} = 1$. An actual average of $\frac{2000}{64} = 31.25$ is $1.25\,\bar{\sigma}$ above the mean. The probability of exceeding $1.25\,\bar{\sigma}$ is about **.1** from Figure 8.12b. With $N = 256$ we still have $\frac{8000}{256} = 31.25$ but now $\bar{\sigma} = \frac{8}{\sqrt{256}} = \frac{1}{2}$. To go $2.5\,\bar{\sigma}$ above the mean has **probability** $< \mathbf{.01}$.

8.5 Masses and Moments (page 340)

1. Masses of 6, 4, and 3 are placed at $x = 4$, $x = 5$, and $x = 8$. Find the *moment* and the *center of mass.*

   - The total mass is $M = 6 + 4 + 3 = 13$. The moment is $m_1x_1 + m_2x_2 + m_3x_3 = 6\cdot 4 + 4\cdot 5 + 3\cdot 8 = 68$. The center of mass is moment divided by total mass, or $\bar{x} = \frac{68}{13} \approx 5.2$.

2. Continuous density along the axis from $x = 0$ to $x = 4$ is given by $\rho = \sqrt{x}$. Find the moment M_y and the center of mass.

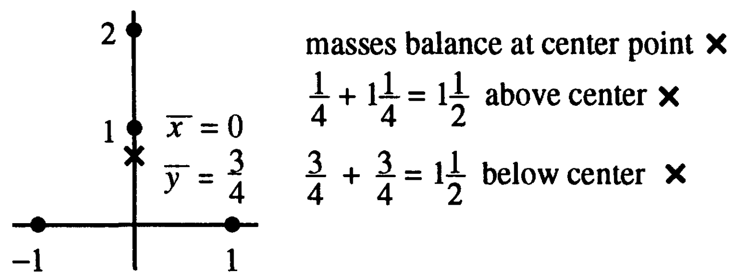
   - Total mass is $M = \int \rho dx = \int_0^4 \sqrt{x}\,dx = \frac{16}{3}$. Total moment is

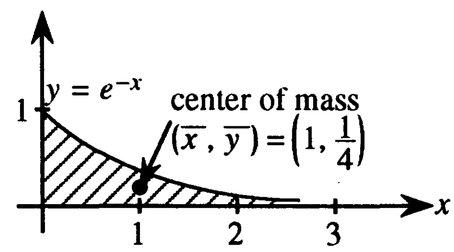
$$M_y = \int_0^4 x\rho(x)dx = \int_0^4 x^{3/2}dx = \frac{2}{5}x^{5/2}]_0^4 = \frac{64}{5}.$$

   The center of mass $\bar{x}$ is $\frac{M_y}{M} = \frac{64/5}{16/3} = \frac{12}{5} = 2.4$.

3. Four unit masses are placed at $(x, y) = (-1,0)$, $(1,0)$, $(0,1)$ and $(0,2)$. Find M_x, M_y, and $(\bar{x}, \bar{y})$.

   - The total mass is $M = 4$. The moment around the y axis is zero because the placement is symmetric. (Check: $1\cdot 1 + 1(-1) + 1\cdot 0 + 1\cdot 0 = 0$.) The moment around the x axis is the sum of the y coordinates (not x!!) times the masses (all 1): $1\cdot 0 + 1\cdot 0 + 1\cdot 1 + 1\cdot 2 = 3$. Then the center of mass is $(\frac{0}{4}, \frac{3}{4}) = (\bar{x}, \bar{y})$.





4. Find the area M and the centroid $(\bar{x}, \bar{y})$ inside the curve $y = e^{-x}$ for $0 \le x < \infty$.

- The area M is the value of the improper integral $\int_0^\infty e^{-x}dx = -e^{-x}]_0^\infty = 1$. For M_y we integrate by parts:

$$M_y = \int_0^\infty xy\,dx = \int_0^\infty xe^{-x}dx = [-xe^{-x} - e^{-x}]_0^\infty = 1.$$

Note: The first term $-xe^{-x}$ approaches zero as $x \to \infty$. We can use l'Hôpital's rule from Section 2.6: $\lim \frac{-x}{e^x} = \lim \frac{-1}{e^x} = 0$. The moment M_x around the x axis is an integral *with respect to y*:

$$M_x = \int_0^1 yx\,dy = \int_0^1 y(-\ln y)dy = [-\frac{1}{2}y^2 \ln y + \frac{1}{4}y^2]_0^1 = \frac{1}{4}.$$

We found $x = -\ln y$ from $y = e^{-x}$. The integration was by parts. At $y = 0$ l'Hôpital's rule evaluates $\lim y^2 \ln y = \lim \frac{\ln y}{y^{-2}} = \lim \frac{\frac{1}{y}}{\frac{-2}{y^3}} = 0$. The centroid is $(\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (1, \frac{1}{4})$.

5. Find the surface area of the cone formed when the line $y = 3x$ from $x = 0$ to $x = 2$ is revolved about the y axis. Use the theorem of Pappus in Problem 8.5.28.

- The Pappus formula $A = 2\pi\bar{y}L$ changes to $A = 2\pi\bar{x}L$, since the rotation is about the y axis and not the x axis. The line goes from (0,0) to (2,6) so its length is $L = \sqrt{2^2 + 6^2} = \sqrt{40}$. Then $\bar{x} = 1$ and the surface area is $A = 2\pi\sqrt{40} = 4\pi\sqrt{10}$.

Read-throughs and selected even-numbered solutions :

If masses m_n are at distances x_n, the total mass is $M = \sum \mathbf{m_n}$. The total moment around $x = 0$ is $M_y = \sum \mathbf{m_n x_n}$. The center of mass is at $\bar{x} = \mathbf{M_y/M}$. In the continuous case, the mass distribution is given by the **density** $\rho(x)$. The total mass is $M = \int \rho(\mathbf{x})\mathbf{dx}$ and the center of mass is at $\bar{x} = \int \mathbf{x\rho(x)dx/M}$. With $\rho = x$, the integrals from 0 to L give $M = \mathbf{L^2/2}$ and $\int x\rho(x)dx = \mathbf{L^3/3}$ and $\bar{x} = \mathbf{2L/3}$. The total moment is the same as if the whole mass M is placed at $\bar{\mathbf{x}}$.

In a plane with masses m_n at the points (x_n, y_n), the moment around the y axis is $\sum \mathbf{m_n x_n}$. The center of mass has $\bar{x} = \sum \mathbf{m_n x_n}/\sum \mathbf{m_n}$ and $\bar{y} = \sum \mathbf{m_n y_n}/\sum \mathbf{m_n}$. For a plate with density $\rho = 1$, the mass M equals the **area**. If the plate is divided into vertical strips of height $y(x)$, then $M = \int y(x)dx$ and $M_y = \int \mathbf{xy(x)}dx$.

For a square plate $0 \le x, y \le L$, the mass is $M = \mathbf{L^2}$ and the moment around the y axis is $M_y = \mathbf{L^3/2}$. The center of mass is at $(\bar{x}, \bar{y}) = (\mathbf{L/2, L/2})$. This point is the **centroid,** where the plate balances.

A mass m at a distance x from the axis has moment of inertia $I = \mathbf{mx^2}$. A rod with $\rho = 1$ from $x = a$ to $x = b$ has $I_y = \mathbf{b^3/3 - a^3/3}$. For a plate with $\rho = 1$ and strips of height $y(x)$, this becomes $I_y = \int \mathbf{x^2 y(x) dx}$. The torque T is **force** times **distance.**

4 $M = \int_0^L x^2 dx = \mathbf{\frac{L^3}{3}}; M_y = \int_0^L x^3 dx = \mathbf{\frac{L^4}{4}}; \bar{x} = \frac{L^4/4}{L^3/3} = \mathbf{\frac{3L}{4}}$.

10 $M = \mathbf{3(\frac{1}{2}ab)}; M_y = \int_0^a 3xb(1 - \frac{x}{a})dx = [\frac{3x^2b}{2} - \frac{x^3b}{a}]_0^a = \mathbf{\frac{a^2b}{2}}$ and by symmetry $M_x = \mathbf{\frac{b^2a}{2}}; \bar{x} = \frac{a^2b/2}{3ab/2} = \mathbf{\frac{a}{3}}$ and $\bar{y} = \mathbf{\frac{b}{3}}$. Note that the centroid of the triangle is at $(\frac{a}{3}, \frac{b}{3}.)$

14 Area $M = \int_0^1 (x - x^2)dx = \mathbf{\frac{1}{6}}; M_y = \int_0^1 x(x - x^2)dx = \frac{1}{12}$ and $\bar{x} = \frac{1/12}{1/6} = \mathbf{\frac{1}{2}}$ (also by symmetry); $M_x = \int_0^1 y(\sqrt{y} - y)dy = \frac{1}{15}$ and $\bar{y} = \frac{1/15}{1/6} = \mathbf{\frac{2}{5}}$.

16 Area $M = \frac{1}{2}(\pi(2)^2 - \pi(0)^2) = \mathbf{\frac{3\pi}{2}}; M_y = 0$ and $\bar{x} = \mathbf{0}$ by symmetry; M_x for halfcircle of radius 2 minus M_x for halfcircle of radius 1 = (by Example 4) $\frac{2}{3}(2^3 - 1^3) = \frac{14}{3}$ and $\bar{y} = \frac{14/3}{3\pi/2} = \mathbf{\frac{28}{9\pi}}$.

18 $I_y = \int_{-a/2}^{a/2} x^2$ (strip height) $dx = \int_{-a/2}^{a/2} x^2 a dx = \frac{a^4}{12}$.

32 Torque $= F - 2F + 3F - 4F \cdots + 9F - 10F = \mathbf{-5F}$.

36 $J = \frac{I}{mr^2}$ is smaller for a solid ball than a solid cylinder because the ball has its mass **nearer the center.**

38 Get most of the mass close to the center but keep the radius large.

42 (a) **False** (a solid ball goes faster than a hollow ball) (b) **False** (if the density is varied, the center of mass moves) (c) **False** (you reduce I_x but you increase I_y : the y direction is upward) (d) **False** (imagine the jumper as an arc of a circle going just over the bar: the center of mass of the arc stays below the bar).

8.6 Force, Work, and Energy (page 346)

1. (This is 8.6.8) The great pyramid outside Cairo (height 500 feet) has a square base 800 feet by 800 feet. If the rock weighs $w = 100$ lb/ft^3, how much work did it take to lift it all?

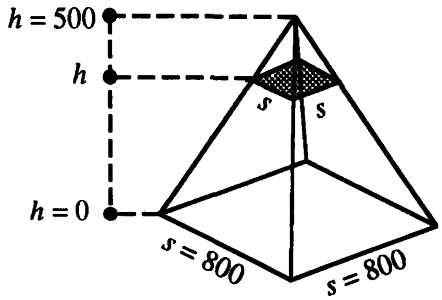
- Imagine the pyramid divided into horizontal square slabs of side s and thickness Δh. The weight of this slab is $ws^2\Delta h$. This slab is carried up to height h. The work done on the slab is therefore $hws^2\Delta h$.

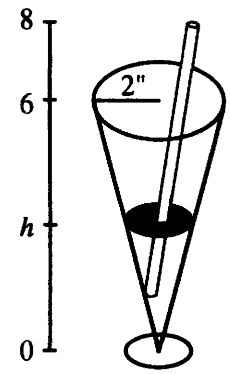
  To get s in terms of h, use a straight line between $s = 800$ when $h = 0$ and $s = 0$ when $h = 500$:

$$s = 800 - \frac{800}{500}h = \frac{8}{5}(500 - h).$$

  Add up the work on all the slabs and let the thickness Δh go to zero (so integrate):

$$W = \int_0^{500} wh(\frac{8}{5}(500 - h))^2 dh = \frac{64}{25}w \int_0^{500} (250,000h - 1000h^2 + h^3)dh = 1.3 \times 10^{12} \text{ ft-lbs}.$$





2. A conical glass is filled with water (6″ high, 2″ radius at the top). How much work is done sipping the water out through a straw? The sipper is 2″ above the glass. Water weighs $w = .036$ lb/in^3.

   - As in the previous problem, calculate the work on a slice of water at height h with thickness Δh. The slice volume is $\pi r^2 \Delta h$ and its weight is $w\pi r^2 \Delta h$. The slice must be lifted $(8 - h)$ inches to get sipped, so the work is $(8 - h)w\pi r^2 \Delta h$. (You may object that water must go down to get into the straw and then be lifted from there. But the net effect of water going down to the bottom and back to its starting level is zero work.) From the shape of the glass, $r = \frac{1}{3}h$. The work integral is

$$w\pi \int_0^6 (8-h)(\frac{h}{3})^2 dh = \frac{w\pi}{9}(\frac{8}{3}h^3 - \frac{1}{4}h^4)]_0^6 = 3.2 \text{ inch-pounds.}$$

3. A 10-inch spring requires a force of 25 pounds to stretch it $\frac{1}{2}$ inch. Find (a) the spring constant k in pounds per foot (b) the work done in stretching the spring (c) the work needed to stretch it an additional half-inch.

   - (a) Hooke's law $F = kx$ gives $k = \frac{F}{x} = \frac{25}{0.5} = 50$ lbs per foot.
   - (b) Work $= \int_0^{1/2} kx\, dx = \frac{1}{2}50x^2]_0^{0.5} = 6.25$ ft-lbs (Not ft/lbs.)
   - (c) Work $= \int_{1/2}^1 kx\, dx = \frac{1}{2}k(1)^2 - \frac{1}{2}k(\frac{1}{2})^2 = 18.75$ ft-lbs. The definite integral is the change in potential energy $\frac{1}{2}kx^2$ between $x = \frac{1}{2}$ and $x = 1$.

4. A rectangular water tank is 100 ft long, 30 ft wide, and 20 ft deep. What is the force on the bottom and on each side of a full tank? Water weighs $w = 62.4$ lb/ft^3.

   - The force on the bottom is $whA = (62.4)(20)(3000) \approx 3.7 \cdot 10^6$ lbs. On the small ends, which are $20' \times 30'$ rectangles, the force is $\int_0^{20} w \cdot 30 \cdot h\, dh$. The limits of integration are 0 to 20 because the water depth h varies from 0 to 20. The value of the integral is $[15wh^2]_0^{20} \approx 3.7 \times 10^4$ lbs. The larger sides are $100' \times 20'$ rectangles, so the force on them is $\int_0^{20} 100wh\, dh \approx 1.2 \times 10^6$ lbs.

Read-throughs and selected even-numbered solutions :

Work equals **force** times **distance**. For a spring the force $F = \mathbf{kx}$ is proportional to the extension x (this is **Hooke's** law). With this variable force, the work in stretching from 0 to x is $W = \int \mathbf{kx\, dx} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{kx^2}$. This equals the increase in the **potential** energy V. Thus W is a **definite** integral and V is the corresponding **indefinite** integral, which includes an arbitrary **constant**. The derivative dV/dx equals **the force**. The force of gravity is $F = \mathbf{GMm/x^2}$ and the potential is $V = \mathbf{-GMm/x}$.

In falling, V is converted to **kinetic** energy $K = \mathbf{\frac{1}{2}mv^2}$. The total energy $K + V$ is **constant** (this is the law of **conservation of energy** when there is no external force).

Pressure is force per unit **area**. Water of density w in a pool of depth h and area A exerts a downward force $F = \mathbf{whA}$ on the base. The pressure is $p = \mathbf{wh}$. On the sides the **pressure** is still wh at depth h, so the total force is $\int whl\ dh$, where l is **the side length at depth h**. In a cubic pool of side s, the force on the base is $F = \mathbf{ws^3}$, the length around the sides is $l = \mathbf{4\pi s}$, and the total force on the four sides is $F = \mathbf{2\pi ws^3}$. The work to pump the water out of the pool is $W = \int whA\ dh = \mathbf{\frac{1}{2}ws^4}$.

2 (a) Spring constant $k = \frac{75 \text{ pounds}}{3 \text{ inches}} = \mathbf{25}$ **pounds per inch**
(b) Work $W = \int_0^3 kxdx = 25(\frac{9}{2}) = \mathbf{\frac{225}{2}}$ **inch-pounds** or $\frac{225}{24}$ foot-pounds (integral starts at no stretch)
(c) Work $W = \int_3^6 kxdx = 25(\frac{36-9}{2}) = \mathbf{\frac{675}{2}}$ **inch-pounds.**

10 The change in $V = -\frac{GmM}{x}$ is $\Delta V = GmM(\frac{1}{R-10} - \frac{1}{R+10}) = GmM\frac{20}{R^2-10^2} = \frac{20GmM}{R^2}\frac{R^2}{R^2-10^2}$. The first factor is the distance (20 feet) times the force (30 pounds). The second factor is the correction (practically 1.)

12 If the rocket starts at R and reaches x, its potential energy increases by $GMm(\frac{1}{R} - \frac{1}{x})$. This equals $\frac{1}{2}mv^2$ (gain in potential = loss in kinetic energy) so $\mathbf{\frac{1}{R} - \frac{1}{x} = \frac{v^2}{2GM}}$ and $\mathbf{x = (\frac{1}{R} - \frac{v^2}{2GM})^{-1}}$. If the rocket reaches $x = \infty$ then $\frac{1}{R} = \frac{v^2}{2GM}$ or $v = \sqrt{\frac{2GM}{R}} = 25{,}000$ mph.

14 A horizontal slice with radius 1 foot, height h feet, and density ρ lbs/ft^3 has potential energy $\pi(1)^2h\rho dh$. Integrate from $h = 0$ to $h = 4$: $\int_0^4 \pi\rho hdh = 8\pi\rho$.

20 Work to empty a cone-shaped tank: $W = \int wAhdh = \int_0^H w\pi r^2\frac{h^3}{H^2}dh = \mathbf{w\pi r^2\frac{H^2}{4}}$. For a cylinder (Problem 17) $W = \frac{1}{2}wAH^2 = w\pi r^2\frac{H^2}{2}$. So the work for a cone is half of the work for a cylinder, even though the volume is only one third. (The cone-shaped tank has more water concentrated near the bottom.)

8 Chapter Review Problems

Review Problems

R1 How do you find the area between the graphs of $y_1(x)$ and $y_2(x)$?

R2 Write the definite integrals for the volumes when an arch of $y = x \sin x$ is revolved about the x axis and the y axis.

R3 Write the definite integrals for the volumes when the region above that arch and below $y = 2$ is revolved about the x axis and y axis.

R4 Write the three formulas for ds. How is ds used in finding the length of a curve and the surface area of a solid of revolution?

R5 What is the difference between discrete and continuous probability? Compare the discrete and continuous formulas for the mean and for the sum of all probabilities.

R6 What happens to the cumulative density function in Figure 8.12 if we let the x axis extend toward $-\infty$ and $+\infty$?

R7 How do the "moments" M_x and M_y differ from the "*moment of inertia*"?

R8 How do you find the work done by a constant force F? By a variable force $F(x)$?

Drill Problems

Find the areas of the regions bounded by the curves in **D1** to **D7**. Answers are not 100% guaranteed.

D1 $y = 4 - x^2, y = x - 2$ Ans $\frac{125}{6}$

D2 $y = x^3, y = x$ Ans $\frac{1}{2}$

D3 $x = \sqrt{y}, x = \sqrt[3]{y}$ Ans $\frac{1}{12}$

D4 $y = x^2, y = \frac{1}{x}, x = 2$ Ans $\frac{7}{13} + \ln 2$

D5 $y = \frac{1}{x^2+4x+5}, x = 0, y = 0$ Ans $\frac{\pi}{2} - \tan^{-1} 2$

D6 $y = \tan^{-1} x, y = \sec^{-1} x, y = 0$ Ans $\ln 2$

D7 $xy = 8, y = 3x - 10, x = 1$ Ans $7.5 + 16 \ln 2$

D8 The region between the parabola $y = x^2$ and the line $y = 4x$ is revolved about the x axis. Find the volume. Ans 8π

D9 The region bounded by $y = x^{-1/2}, x = \frac{1}{4}, x = 1$, and $y = 0$ is revolved about the x axis. Find the volume. Ans $2\pi \ln 2$

D10 The region between $y = e^{x/2}, x = 0$, and $y = e$ is revolved about the x axis. Find $\pi(e^2 + 1)$ as volume.

D11 The region bounded by $y = x \ln x, x = 0, x = \pi$, and the x axis is revolved about the y axis. Find the volume. Ans $2\pi^2$

D12 The region bounded by $y = \sin x^2, y = 0$, and $x = \sqrt{\pi/2}$ is revolved about the y axis. Find the volume. Ans π

D13 The right half of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is revolved about the y axis. Find the volume. Ans 16π

D14 A chopped-off cone is $15'$ high. The upper radius is $3'$ while the lower radius is $10'$. Find its volume by horizontal slices. Ans 695π

D15 Find the length of $y = \frac{x^4}{8} + \frac{1}{4x^2}$ from $x = 1$ to $x = 2$. Ans $\frac{17}{12}$

D16 Find the length of $y = 2x^{2/3}$ from $x = 1$ to $x = 8$. Ans $\frac{17}{12}$

D17 Set up an integral for the length of $y = \ln x$ from $x = 1$ to $x = e$.

D18 Find the length of the curve $e^x = \sin y$ for $\frac{\pi}{6} \leq y \leq \frac{\pi}{2}$. Ans $\ln(2 + \sqrt{3})$

D19 Find the length of the curve $x = \frac{1}{2}\ln(1 + t^2),\ y = \tan^{-1} t$ from $t = 0$ to $t = \pi$. Ans $\sqrt{2}(e^\pi - 1)$

D20 Find the length of the curve $x = \frac{1}{2}t^2,\ y = \frac{1}{15}(10t + 25)^{3/2}$ from $t = 0$ to $t = 4$. Ans 28

D21 Find the length of the curve $x = t^{3/2},\ y = 4t,\ 0 \leq t \leq 4$. Ans $\frac{488}{27}$

Find the areas of the surfaces formed by revolving the curves **D22** to **D26**.

D22 $y = x^3$ around the x axis, $0 \leq x \leq 2$ Ans $\frac{\pi}{27}(145^{3/2} - 1)$

D23 $x = 2\sqrt{y}$ around the y axis, $1 \leq y \leq 4$ Ans $\frac{8\pi}{3}(5^{3/2} - 2^{3/2})$

D24 $y = \sin x^2$ around the x axis, $0 \leq x \leq \frac{\sqrt{\pi}}{2}$ Ans $\frac{\pi}{2}(2 - \sqrt{2})$

D25 $y = \frac{x^3}{6} + \frac{1}{2x}$ around the y axis, $1 \leq x \leq 2$ Ans $\pi(3.75 + \ln 2)$

D26 $x = \ln\sqrt{y} - \frac{1}{4}y^2$ around the y axis, $0 \le y \le 3$ Ans $\frac{32\pi}{3}$

D27 The line at a post office has the distribution $p(x) = 0.4e^{-0.4x}$, where x is the waiting time in minutes for a random customer. (a) What is the mean waiting time? (b) What percentage of customers wait longer than six minutes? Ans (a) 4 min (b) $e^{-2.4} \approx 9.1\%$

D28 The weights of oranges vary normally with a mean of 5 ounces and a standard deviation of 9.8 ounces. 95% of the oranges lie between what weights? Ans 3.4 oz and 6.6 oz

D29 Show that $M_x = \frac{1}{10}, M_y = \frac{1}{4}$, and $(\bar{x}, \bar{y}) = (\frac{3}{4}, \frac{3}{10})$ for the region between $y = x^2, y = 0$, and $x = 1$.

D30 Find the moments of inertia I_x and I_y for the triangle with corners (0,0), (2,0) and (0,3).

D31 A force of 50 pounds stretches a spring 4 inches. Find the work required to stretch it an additional 4 inches. Ans 25 ft - lbs.

D32 A 10′ hanging rope weighs 2 lbs. How much work to wind up the rope? Ans 10 ft - lbs.

Calculator Problems

C1 Find the length of the graph $y = x^3$ from (0,0) to (2,8). Ans 8.3

C2 Compute the surface area when the arc $x^2 + y^2 = 25$ from (3,4) to (4,3) is revolved about the x axis.

C3 Find the distance around the ellipse $x^2 + 4y^2 = 1$.