CHAPTER 7 TECHNIQUES OF INTEGRATION

7.1 Integration by Parts (page 287)

Integration by parts aims to exchange a difficult problem for a possibly longer but probably easier one. It is up to you to make the problem easier! The key lies in choosing "u" and "dv" in the formula $\int u\,dv = uv - \int v\,du$. Try to pick u so that du is simple (or at least no worse than u). For $u = x$ or x^2 the derivative 1 or $2x$ is simpler. For $u = \sin x$ or $\cos x$ or e^x it is no worse. On the other hand, choose "dv" to have a nice integral. Good choices are $dv = \sin x\,dx$ or $\cos x\,dx$ or $e^x\,dx$.

Of course the selection of u also decides dv (since $u\,dv$ is the given integration problem). Notice that $u = \ln x$ is a good choice because $du = \frac{1}{x}dx$ is simpler. On the other hand, $\ln x\,dx$ is usually a poor choice for dv, because its integral $x\ln x - x$ is more complicated. Here are more suggestions:

Good choices for u: $\ln x$, inverse trig functions, $x^n, \cos x, \sin x, e^x$ or e^{cx}.

These are just suggestions. It's a free country. Integrate $1 - 6$ by parts:

1. $\int xe^{-x}dx$.

- Pick $u = x$ because $\frac{du}{dx} = 1$ is simpler. Then $dv = e^{-x}dx$ gives $v = -e^{-x}$. Watch all the minus signs:

$$\int u\,dv = \underset{\boldsymbol{u}}{x}\ \underset{\boldsymbol{v}}{(-e^{-x})} - \int \underset{\boldsymbol{v}}{(-e^{-x})}\ \underset{\boldsymbol{du}}{dx} = -xe^{-x} - e^{-x} + C$$

2. $\int x\sec^{-1} x\,dx$.

- If we choose $u = x$, we are faced with $dv = \sec^{-1} x\,dx$. Its integral is difficult. Better to try $u = \sec^{-1} x$, so that $du = \frac{dx}{|x|\sqrt{x^2-1}}$. Is that simpler? It leaves $dv = x\,dx$, so that $v = \frac{1}{2}x^2$. Our integral is now $uv - \int v\,du$:

$$(\sec^{-1} x)(\tfrac{1}{2}x^2) - \int \tfrac{1}{2}x^2 \cdot \frac{dx}{|x|\sqrt{x^2-1}} = \tfrac{1}{2}x^2 \sec^{-1} x \pm \tfrac{1}{2}\int \frac{x\,dx}{\sqrt{x^2-1}}$$
$$= \tfrac{1}{2}x^2 \sec^{-1} x \pm \tfrac{1}{2}\sqrt{x^2-1} + C.$$

The $\pm$ sign comes from $|x|$; plus if $x > 0$ and minus if $x < 0$.

3. $\int e^x \sin x\,dx$. (Problem 7.1.9) This example requires *two* integrations by parts. First choose $u = e^x$ and $dv = \sin x\,dx$. This makes $du = e^x\,dx$ and $v = -\cos x$. The first integration by parts is $\int e^x \sin x\,dx = -e^x\cos x + \int e^x \cos x\,dx$. The new integral on the right is no simpler than the old one on the left. For the new one, $dv = \cos x\,dx$ brings back $v = \sin x$:

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

Are we back where we started? Not quite. Put the second into the first:

$$\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx.$$

The integrals are now the same. Move the one on the right side to the left side, *and divide by* 2:

$$\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$

4. $\int x^2 \ln x \, dx$ (Problem 7.1.6). The function $\ln x$ (if it appears) is almost always the choice for u. Then $du = \frac{dx}{x}$. This leaves $dv = x^2 dx$ and $v = \frac{1}{3}x^3$. Therefore

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

5. $\int \frac{x^3 dx}{\sqrt{x^2+1}}$.

- Generally we choose u for a nice derivative, and dv is what's left. In this case it pays for dv to have a nice integral. We don't know $\int \frac{x^3}{\sqrt{x^2+1}}dx$ but we do know $\int \frac{x}{\sqrt{x^2+1}}dx = \sqrt{x^2+1}$. This leaves $u = x^2$ with $du = 2x \, dx$:

$$\begin{aligned} \int \frac{x^3 dx}{\sqrt{x^2+1}} &= x^2\sqrt{x^2+1} - \int 2x\sqrt{x^2+1} \, dx \\ &= x^2\sqrt{x^2+1} - \tfrac{2}{3}(x^2+1)^{3/2} + C. \end{aligned}$$

Note Integration by parts is not the only way to do this problem. You can directly substitute $u = x^2 + 1$ and $du = 2x \, dx$. Then x^2 is $u - 1$ and $x \, dx$ is $\frac{1}{2}du$. The integral is

$$\begin{aligned} \frac{1}{2}\int \frac{u-1}{\sqrt{u}}du &= \frac{1}{2}\int (u^{1/2} - u^{-1/2})du = \frac{1}{3}u^{3/2} - u^{1/2} + C \\ &= \frac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C \quad \text{(same answer in disguise).} \end{aligned}$$

6. Derive a reduction formula for $\int (\ln x)^n dx$.

- A reduction formula gives this integral in terms of an integral of $(\ln x)^{n-1}$. Let $u = (\ln x)^n$ so that $du = n(\ln x)^{n-1}(\frac{1}{x})dx$. Then $dv = dx$ gives $v = x$. This cancels the $\frac{1}{x}$ in du :

$$\int (\ln x)^n dx = x(\ln x)^n - \int n(\ln x)^{n-1} dx.$$

6′. Find a similar reduction from $\int x^n e^x \, dx$ to $\int x^{n-1} e^x \, dx$.

7. Use this reduction formula as often as necessary to find $\int (\ln x)^3 dx$.

- Start with $n = 3$ to get $\int (\ln x)^3 dx = x(\ln x)^3 - 3\int (\ln x)^2 dx$. Now use the formula with $n = 2$. The last integral is $x(\ln x)^2 - 2\int \ln x \, dx$. Finally $\int \ln x \, dx$ comes from $n = 1$: $\int \ln x \, dx = x(\ln x) - \int (\ln x)^0 dx = x(\ln x) - x$. Substitute everything back:

$$\begin{aligned} \int (\ln x)^3 dx &= x(\ln x)^3 - 3[x(\ln x)^2 - 2[x \ln x - x]] + C \\ &= x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C. \end{aligned}$$

Problems 8 and 9 are about the step function $U(x)$ and its derivative the delta function $\delta(x)$.

8. Find $\int_{-2}^{6} (x^2 - 8)\delta(x) dx$

- Since $\delta(x) = 0$ everywhere except at $x = 0$, we are only interested in $v(x) = x^2 - 8$ at $x = 0$. At that point $v(0) = -8$. We separate the problem into two parts:

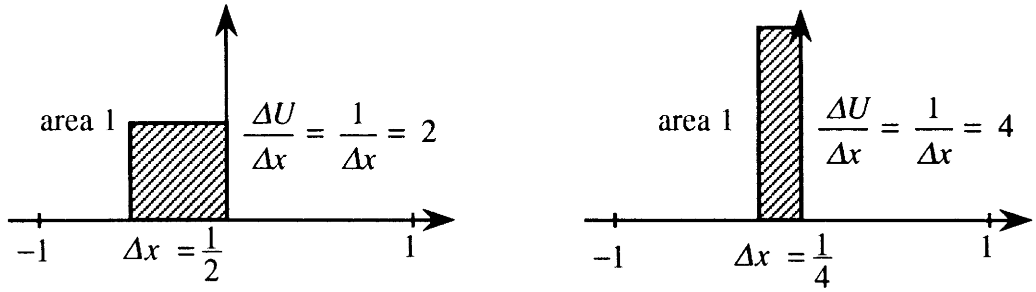
$$\int_{-2}^{2} (x^2 - 8)\delta(x) dx + \int_{2}^{6} (x^2 - 8)\delta(x) dx = -8 + 0.$$

The first integral is just like 7B, picking out $v(0)$. The second integral is zero since $\delta(x) = 0$ in the interval [2,6]. The answer is -8.

9. (This is 7.1.54) Find the area under the graph of $\frac{\Delta U}{\Delta x} = [U(x+\Delta x) - U(x)]/\Delta x$.

- For the sake of this discussion let Δx be positive. The step function has $U(x) = 1$ if $x \geq 0$. In that case $U(x+\Delta x) = 1$ also. Subtraction $U(x+\Delta x) - U(x)$ leaves zero. The only time $U(x+\Delta x)$ is different from $U(x)$ is when $x + \Delta x \geq 0$ and $x < 0$. In that case

$$U(x+\Delta x) - U(x) = 1 - 0 = 1 \text{ and } \frac{U(x+\Delta x) - U(x)}{\Delta x} = \frac{1}{\Delta x}.$$



The sketches show the small interval $-\Delta x \leq x < 0$ where this happens. The base of the rectangle is Δx but the height is $\frac{1}{\Delta x}$. ***The area stays constant at 1.***

The limit of $\frac{U(x+\Delta x)-U(x)}{\Delta x}$ is the slope of the step function. This is the delta function $U'(x) = \delta(x)$. Certainly $\delta(x) = 0$ except at $x = 0$. But the integral of the delta function across the spike at $x = 0$ is 1. (The area hasn't changed as $\Delta x \to 0$.) A strange function.

Read-throughs and selected even-numbered solutions :

Integration by parts is the reverse of the **product** rule. It changes $\int u\,dv$ into **uv** minus $\int$ **v du**. In case $u = x$ and $dv = e^{2x}dx$, it changes $\int xe^{2x}dx$ to $\mathbf{\frac{1}{2}xe^{2x}}$ minus $\int$ $\mathbf{\frac{1}{2}\,e^{2x}dx}$. The definite integral $\int_0^2 xe^{2x}dx$ becomes $\mathbf{\frac{3}{4}e^4}$ minus $\mathbf{\frac{1}{4}}$. In choosing u and dv, the **derivative** of u and the **integral** of dv/dx should be as simple as possible. Normally $\ln x$ goes into **u** and e^x goes into **v**. Prime candidates are $u = x$ or x^2 and $v = \sin x$ or $\mathbf{\cos x}$ or $\mathbf{e^x}$. When $u = x^2$ we need **two** integrations by parts. For $\int \sin^{-1} x\,dx$, the choice $dv = dx$ leads to $\mathbf{x \sin^{-1} x}$ minus $\int \mathbf{x\,dx/\sqrt{1-x^2}}$.

If U is the unit step function, $dU/dx = \delta$ is the unit **delta** function. The integral from $-A$ to A is $U(A) - U(-A) = \mathbf{1}$. The integral of $v(x)\delta(x)$ equals $\mathbf{v(0)}$. The integral $\int_{-1}^{1} \cos x\ \delta(x)dx$ equals **1**. In engineering, the balance of forces $-dv/dx = f$ is multiplied by a displacement $u(x)$ and integrated to give a balance of **work.**

14 $\int \cos(\ln x)dx = uv - \int v du = \cos(\ln x)x + \int x \sin(\ln x)\frac{1}{x}dx$. Cancel x with $\frac{1}{x}$. Integrate by parts again to get $\cos(\ln x)x + \sin(\ln x)x - \int x\cos(\ln x)\frac{1}{x}dx$. Move the last integral to the left and divide by 2.
The answer is $\frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$.

18 $uv - \int v\,du = \cos^{-1}(2x)x + \int x\frac{2\,dx}{\sqrt{1-(2x)^2}} = x\cos^{-1}(2x) - \frac{1}{2}(1-4x^2)^{1/2} + C$.

22 $uv - \int v\,du = x^3(-\cos x) + \int(\cos x)3x^2dx =$ (use Problem 5) $= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$.

28 $\int_0^1 e^{\sqrt{x}}dx = \int_{u=0}^1 e^u(2u\,du) = 2e^u(u-1)]_0^1 = \mathbf{2}$. **38** $\int x^n \sin x\,dx = -x^n\cos x + n\int x^{n-1}\cos x\,dx$.

44 (a) $e^0 = \mathbf{1}$; (b) $\mathbf{v(0)}$ (c) **0** (limits do not enclose zero).

46 $\int_{-1}^1 \delta(2x)dx = \int_{u=-2}^2 \delta(u)\frac{du}{2} = \mathbf{\frac{1}{2}}$. Apparently $\delta(2x)$ equals $\frac{1}{2}\delta(x)$; both are zero for $x \neq 0$.

48 $\int_0^1 \delta(x-\frac{1}{2})dx = \int_{-1/2}^{1/2}\delta(u)du = \mathbf{1}$; $\int_0^1 e^x\delta(x-\frac{1}{2})dx = \int_{-1/2}^{1/2} e^{u+\frac{1}{2}}\delta(u)du = \mathbf{e^{1/2}}$; $\delta(x)\delta(x-\frac{1}{2}) = \mathbf{0}$.

60 $A = \int_1^e \ln x\,dx = [x\ \ln x - x]_1^e = \mathbf{1}$ is the area under $y = \ln x$. $B = \int_0^1 e^y dy = \mathbf{e-1}$ is the area to the left of $y = \ln x$. Together the area of the rectangle is $1 + (e-1) = e$.

7.2 Trigonometric Integrals (page 293)

This section integrates powers and products of sines and cosines and tangents and secants. We are constantly using $\sin^2 x = 1 - \cos^2 x$. Starting with $\int \sin^3 x\, dx$, we convert it to $\int (1 - \cos^2 x)\sin x\, dx$. Are we unhappy about that one remaining $\sin x$? *Not at all.* It will be part of du, when we set $u = \cos x$. Odd powers are actually easier than even powers, because the extra term goes into du. For even powers use the double-angle formula in Problem 2 below.

1. $\int (\sin x)^{-3/2}(\cos x)^3 dx$ is a product of sines and cosines.

   - The angles x are the same and *the power* 3 *is odd.* ($-\frac{3}{2}$ is neither even nor odd.) Change all but one of the cosines to sines by $\cos^2 x = 1 - \sin^2 x$. The problem is now

   $$\int (\sin x)^{-3/2}(1 - \sin^2 x)\cos x\, dx = \int (u^{-3/2} - u^{1/2})du.$$

   Here $u = \sin x$ and $du = \cos x\, dx$. The answer is

   $$-2u^{-1/2} - \frac{2}{3}u^{3/2} + C = -2(\sin x)^{-1/2} - \frac{2}{3}(\sin x)^{3/2} + C.$$

2. $\int \sin^4 3x\ \cos^2 3x\ dx$ has even powers 4 and 2, with the same angle $3x$.

   - *Use the double-angle method.* Replace $\sin^2 3x$ with $\frac{1}{2}(1 - \cos 6x)$ and $\cos^2 3x$ with $\frac{1}{2}(1 + \cos 6x)$. The problem is now

   $$\begin{aligned}\int \frac{(1-\cos 6x)^2}{4}\frac{(1+\cos 6x)}{2}dx &= \tfrac{1}{8}\int (1 - 2\cos 6x + \cos^2 6x)(1 + \cos 6x)dx \\ &= \tfrac{1}{8}\int (1 - \cos 6x - \cos^2 6x + \cos^3 6x)dx.\end{aligned}$$

   The integrals of the first two terms are x and $\frac{1}{6}\sin 6x$. The third integral is another double angle:

   $$\int \cos^2 6x\ dx = \int \frac{1}{2}(1 + \cos 12x)dx = \frac{1}{2}x + \frac{1}{24}\sin 12x.$$

   For $\int \cos^3 6x\ dx$, with an odd power, change $\cos^2$ to $1 - \sin^2$:

   $$\int \cos^3 6x\ dx = \int (1 - \sin^2 6x)\cos 6x\ dx = \int (1 - u^2)\frac{du}{6} = \frac{1}{6}\sin 6x - \frac{1}{18}\sin^3 6x.$$

   Putting all these together, the final solution is

   $$\frac{1}{8}[x - \frac{1}{6}\sin 6x - (\frac{1}{2}x + \frac{1}{24}\sin 12x) + \frac{1}{6}\sin 6x - \frac{1}{18}\sin^3 6x] = \frac{1}{16}x - \frac{1}{192}\sin 12x - \frac{1}{144}\sin^3 6x + C.$$

3. $\int \sin 10x\ \cos 4x\ dx$ has different angles $10x$ and $4x$. Use the identity $\sin 10x \cos 4x = \frac{1}{2}\sin(10 + 4)x + \frac{1}{2}\sin(10 - 4)x$. Now integrate:

   $$\int (\frac{1}{2}\sin 14x + \frac{1}{2}\sin 6x)dx = -\frac{1}{28}\cos 14x - \frac{1}{12}\cos 6x + C.$$

4. $\int \cos x \cos 4x \cos 8x \, dx$ has three different angles!

- Use the identity $\cos 4x \cos 8x = \frac{1}{2}\cos(4+8)x + \frac{1}{2}\cos(4-8)x$. The integral is now $\frac{1}{2}\int(\cos x \cos 12x + \cos x \cos 4x)dx$. Apply the $\cos px \cos qx$ identity twice more to get

$$\frac{1}{2}\int\left(\frac{1}{2}\cos 13x + \frac{1}{2}\cos 11x + \frac{1}{2}\cos 5x + \frac{1}{2}\cos 3x\right)dx = \frac{1}{4}\left(\frac{\sin 13x}{13} + \frac{\sin 11x}{11} + \frac{\sin 5x}{5} + \frac{\sin 3x}{3}\right) + C.$$

5. $\int \tan^5 x \, \sec^4 x \, dx$. Here are three ways to deal with tangents and secants.

- *First*: Remember $d(\tan x) = \sec^2 x \, dx$ and convert the other $\sec^2 x$ to $1 + \tan^2 x$. The problem is

$$\int \tan^5 x(1+\tan^2 x)\sec^2 x \, dx = \int (u^5 + u^7)du.$$

- *Second*: Remember $d(\sec x) = \sec x \tan x \, dx$ and convert $\tan^4 x$ to $(\sec^2 x - 1)^2$. The integral is

$$\int (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x \, dx = \int (u^2-1)^2 u^3 du = \int (u^7 - 2u^5 + u^3)du.$$

- *Third*: Convert $\tan^5 x \sec^4 x$ to sines and cosines as $\frac{\sin^5 x}{\cos^9 x}$. Eventually take $u = \cos x$:

$$\begin{aligned} \int \frac{(1-\cos^2 x)^2}{\cos^9 x}\sin x \, dx &= \int(\cos^{-9} x - 2\cos^{-7} x + \cos^{-5} x)\sin x \, dx \\ &= \int(-u^{-9} + 2u^{-7} - u^{-5})du. \end{aligned}$$

6. Use the substitution $u = \tan\frac{x}{2}$ in the text equation (11) to find $\int_0^{\pi/4} \frac{dx}{1-\sin x}$.

- The substitutions are $\sin x = \frac{2u}{1+u^2}$ and $dx = \frac{2du}{1+u^2}$. This gives

$$\int \frac{1}{1-\sin x}dx = \int \frac{1}{1-\frac{2u}{1+u^2}} \cdot \frac{2du}{1+u^2} = \int \frac{2du}{(1+u^2)-2u} = \int \frac{2du}{(1-u)^2} = \frac{2}{1-u}.$$

The definite integral is from $x = 0$ to $x = \frac{\pi}{4}$. Then $u = \tan\frac{x}{2}$ goes from 0 to $\tan\frac{\pi}{8}$. The answer is $\frac{2}{1-\tan\frac{\pi}{8}} - \frac{2}{1} \approx 1.41$.

7. Problem 7.2.26 asks for $\int_0^\pi \sin 3x \sin 5x \, dx$. First write $\sin 3x \sin 5x$ in terms of $\cos 8x$ and $\cos 2x$.

- The formula for $\sin px \sin qx$ gives

$$\int_0^\pi \left(-\frac{1}{2}\cos 8x + \frac{1}{2}\cos 2x\right)dx = \left[-\frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x\right]_0^\pi = 0.$$

8. Problem 7.2.33 is the Fourier sine series $A\sin x + B\sin 2x + C\sin 3x + \cdots$ that adds to x. Find A.

- Multiply both sides of $x = A\sin x + B\sin 2x + C\sin 3x + \cdots$ by $\sin x$. *Integrate from* 0 *to* π:

$$\int_0^\pi x\sin x \, dx = \int_0^\pi A\sin^2 x \, dx + \int_0^\pi B\sin 2x \ \sin x \, dx + \int_0^\pi C\sin 3x \sin x \, dx + \cdots.$$

All of the definite integrals on the right are zero, except for $\int_0^\pi A\sin^2 x \, dx$. For example the integral of $\sin 2x \sin x$ is $[-\frac{1}{6}\sin 3x + \frac{1}{2}\sin x]_0^\pi = 0$. The only nonzero terms are $\int_0^\pi x\sin x \, dx = \int_0^\pi A\sin^2 x \, dx$. Integrate $x\sin x$ by parts to find one side of this equation for A :

$$\int_0^\pi x\sin x \, dx = [-x\cos x]_0^\pi + \int_0^\pi \cos x \, dx = [-x\cos x + \sin x]_0^\pi = \pi.$$

On the other side $\int_0^\pi A\sin^2 x \, dx = \frac{A}{2}[x - \sin x\cos x]_0^\pi = \frac{A\pi}{2}$. Then $\frac{A\pi}{2} = \pi$ and $A = 2$.

- You should *memorize* those integrals $\int_0^\pi \sin^2 x dx = \int_0^\pi \cos^2 x\, dx = \frac{\pi}{2}$. They say that the average value of $\sin^2 x$ is $\frac{1}{2}$, and the average value of $\cos^2 x$ is $\frac{1}{2}$.
- You would find B by multiplying the Fourier series by $\sin 2x$ instead of $\sin x$. This leads in the same way to $\int_0^\pi x \sin 2x\, dx = \int_0^\pi B \sin^2 2x\, dx = B\frac{\pi}{2}$ because all other integrals are zero.

9. When a sine and a cosine are added, the resulting wave is best expressed as a single cosine: $a\cos x + b\sin x = \sqrt{a^2+b^2}\cos(x-\alpha)$. Show that this is correct and find the angle α (Problem 7.2.56).

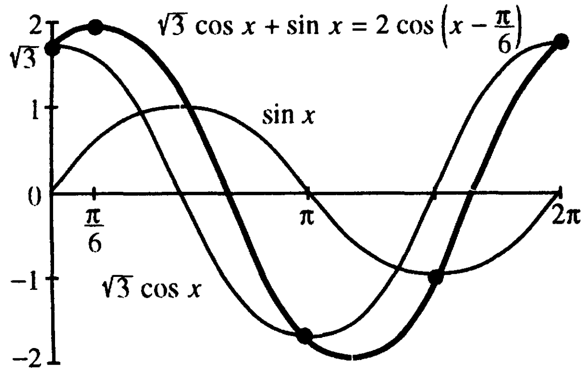
- Expand $\cos(x-\alpha)$ into $\cos x\cos\alpha + \sin x\sin\alpha$. Choose α so that $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$. Our formula becomes correct. The reason for $\sqrt{a^2+b^2}$ is to ensure that $\cos^2\alpha + \sin^2\alpha = \frac{a^2+b^2}{a^2+b^2} = 1$. Dividing $\sin\alpha$ by $\cos\alpha$ gives $\tan\alpha = \frac{b}{a}$ or $\alpha = \tan^{-1}\frac{b}{a}$. Thus $3\cos x + 4\sin x = 5\cos(x - \tan^{-1}\frac{4}{3})$.

10. Use the previous answer (Problem 9) to find $\int \frac{dx}{\sqrt{3}\cos x + \sin x}$.

- With $a = \sqrt{3}$ and $b = 1$ we have $\sqrt{a^2+b^2} = \sqrt{3+1} = 2$ and $\alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. Therefore

$$\int \frac{dx}{\sqrt{3}\cos x + \sin x} = \int \frac{dx}{2\cos(x-\pi/6)} = \frac{1}{2}\int \sec(x-\frac{\pi}{6})dx = \frac{1}{2}\ln|\sec(x-\frac{\pi}{6}) + \tan(x-\frac{\pi}{6})| + C.$$

The figure shows the waves $\sqrt{3}\cos x$ and $\sin x$ adding to $2\cos(x-\frac{\pi}{6})$.



11. What is the distance from the equator to latitude 45° on a Mercator world map? From 45° to 70°?

- The distance north is the integral of $\sec x$, multiplied by the radius R of the earth (on your map). See Figure 7.3 in the text. The equator is at 0°. The distance to 45° = $\frac{\pi}{4}$ radians is

$$R\int_0^{\pi/4} \sec x\, dx = R\ln(\sec x + \tan x)_0^{\pi/4} = R\ln(\sqrt{2}+1) - R\ln 1 \approx 0.88R.$$

The distance from 45° to 70° is almost the same: $R\ln|\sec x + \tan x|\Big|_{45^\circ}^{70^\circ} \approx 0.85R$.

Read-throughs and selected even-numbered solutions :

To integrate $\sin^4 x \cos^3 x$, replace $\cos^2 x$ by $\mathbf{1 - \sin^2 x}$. Then $(\sin^4 x - \sin^6 x)\cos x\, dx$ is $(\mathbf{u^4 - u^6})du$. In terms of $u = \sin x$ the integral is $\mathbf{\frac{1}{5}u^5 - \frac{1}{7}u^7}$. This idea works for $\sin^m x \cos^n x$ if m or n is **odd**.

If both m and n are **even**, one method is integration by **parts**. For $\int \sin^4 x\, dx$, split off $dv = \sin x\, dx$.

Then $-\int v\,du$ is $\int \mathbf{3\sin^2x\cos^2x}$. Replacing $\cos^2 x$ by $\mathbf{1-\sin^2x}$ creates a new $\sin^4 x\,dx$ that combines with the original one. The result is a *reduction* to $\int \sin^2 x\,dx$, which is known to equal $\mathbf{\frac{1}{2}(x-\sin x\cos x)}$.

The second method uses the double-angle formula $\sin^2 x = \mathbf{\frac{1}{2}(1-\cos 2x)}$. Then $\sin^4 x$ involves $\cos^2 \mathbf{2x}$. Another doubling comes from $\cos^2 2x = \mathbf{\frac{1}{2}(1+\cos 4x)}$. The integral contains the sine of $\mathbf{4x}$.

To integrate $\sin\,6x\cos\,4x$, rewrite it as $\frac{1}{2}\sin 10x + \mathbf{\frac{1}{2}\sin 2x}$. The integral is $\mathbf{-\frac{1}{20}\cos 10x - \frac{1}{4}\cos 2x}$. The definite integral from 0 to 2π is **zero**. The product $\cos px\ \cos qx$ is written as $\frac{1}{2}\cos(p+q)x + \mathbf{\frac{1}{2}\cos(p-q)x}$. Its integral is also zero, except if $\mathbf{p=q}$ when the answer is π.

With $u=\tan x$, the integral of $\tan^9 x\ \sec^2 x$ is $\mathbf{\frac{1}{10}\tan^{10}x}$. Similarly $\int \sec^9 x(\sec\ x\ \tan\ x\,dx) = \mathbf{\frac{1}{10}\sec^{10}x}$. For the combination $\tan^m x\ \sec^n x$ we apply the identity $\tan^2 x = \mathbf{1+\sec^2 x}$. After reduction we may need $\int \tan\ x\,dx = \mathbf{-\ln\cos x}$ and $\int \sec\ x\,dx = \mathbf{\ln(\sec x+\tan x)}$.

6 $\int \sin^3 x\cos^3 x\,dx = \int \sin^3 x(1-\sin^2 x)\cos x\,dx = \mathbf{\frac{1}{4}\sin^4x - \frac{1}{6}\sin^6x} + C$

10 $\int \sin^2 ax\cos ax\,dx = \mathbf{\frac{\sin^3 ax}{3a}} + C$ and $\int \sin ax\cos ax\,dx = \mathbf{\frac{\sin^2 ax}{2a}} + C$

16 $\int \sin^2 x\cos^2 2x\,dx = \int \frac{1-\cos 2x}{2}\cos^2 2x\,dx = \int(\frac{1+\cos 4x}{4} - \frac{\cos 2x}{2}(1-\sin^2 2x))dx =$
$\mathbf{\frac{x}{4} + \frac{\sin 4x}{16} - \frac{\sin 2x}{4} + \frac{\sin^3 2x}{12}} + C$. This is a hard one.

18 Equation (7) gives $\int_0^{\pi/2}\cos^n x\,dx = [\frac{\cos^{n-1}x\sin x}{n}]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2} x\,dx$. The integrated term is zero because $\cos\frac{\pi}{2}=0$ and $\sin 0 = 0$. The exception is $n=1$, when the integral is $[\sin x]_0^{\pi/2} = 1$.

26 $\int_0^\pi \sin 3x\sin 5x\,dx = \int_0^\pi \frac{-\cos 8x+\cos 2x}{2}dx = [\mathbf{\frac{-\sin 8x}{16} + \frac{\sin 2x}{4}}]_0^\pi = \mathbf{0}$.

30 $\int_0^{2\pi}\sin x\sin 2x\sin 3x\,dx = \int_0^{2\pi}\sin 2x(\frac{-\cos 4x+\cos 2x}{2})dx = \int_0^{2\pi}\sin 2x(\frac{1-2\cos^2 2x+\cos\ 2x}{2})dx =$
$[\mathbf{-\frac{\cos 2x}{4} + \frac{\cos^3 2x}{6} - \frac{\cos^2 2x}{8}}]_0^{2\pi} = \mathbf{0}$. Note: The integral has other forms.

32 $\int_0^\pi x\ \cos\ x\,dx = [x\ \sin\ x]_0^\pi - \int_0^\pi \sin x\,dx = [\mathbf{x\sin x+\cos x}]_0^\pi = \mathbf{-2}$.

34 $\int_0^\pi 1\sin 3x\,dx = \int_0^\pi(A\sin x + B\sin 2x + C\sin 3x+\cdots)\sin 3x\,dx$ reduces to $[-\frac{\cos 3x}{3}]_0^\pi = 0+0+C\int_0^\pi \sin^2 3x\,dx$. Then $\frac{2}{3} = C(\frac{\pi}{2})$ and $\mathbf{C=\frac{4}{3\pi}}$.

44 First by substituting for $\tan^2 x$: $\int \tan^2 x\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$. Use Problem 62 to integrate $\sec^3 x$: final answer $\mathbf{\frac{1}{2}(\sec x\tan x - \ln|\sec x+\tan x|)} + C$. Second method from line 1 of Example 11: $\int \tan^2 x\sec x\,dx = \sec x\tan x - \int \sec^3 x\,dx$. Same final answer.

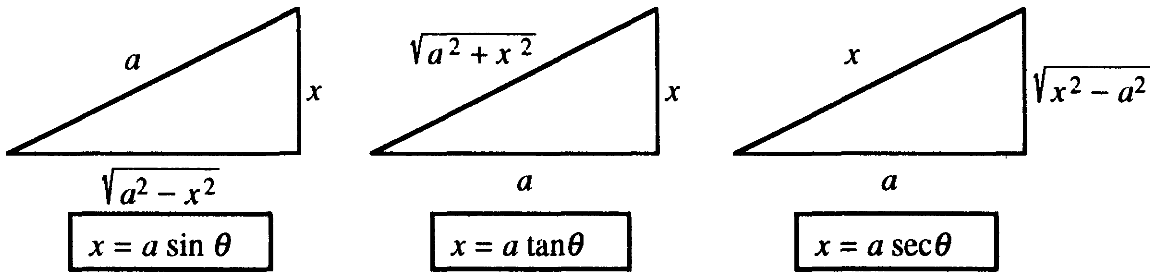
52 This should have an asterisk! $\int \frac{\sin^6 x}{\cos^3 x}dx = \int \frac{(1-\cos^2 x)^3}{\cos^3 x}dx = \int(\sec^3 x - 3\sec x + 3\cos x - \cos^3 x)dx =$ use Example 11 = Problem 62 for $\int \sec^3 x\,dx$ and change $\int \cos^3 x\,dx$ to $\int(1-\sin^2 x)\cos x\,dx$.
Final answer $\mathbf{\frac{\sec x\tan x}{2} - \frac{5}{2}\ln|\sec x+\tan x| + 2\sin x + \frac{\sin^3 x}{3}} + C$.

54 $\mathbf{A=2}$: $2\cos(x+\frac{\pi}{3}) = 2\cos x\cos\frac{\pi}{3} - 2\sin x\sin\frac{\pi}{3} = \cos x - \sqrt{3}\sin x$. Therefore $\int \frac{dx}{(\cos x-\sqrt{3}\sin x)^2} =$
$\int \frac{dx}{4\cos^2(x+\frac{\pi}{3})} = \mathbf{\frac{1}{4}\tan(x+\frac{\pi}{3})} + C$.

58 When lengths are scaled by $\sec x$, area is scaled by $\mathbf{\sec^2x}$. The area from the equator to latitude x is then proportional to $\int \sec^2 x\,dx = \tan x$.

7.3 Trigonometric Substitutions (page 299)

The substitutions may be easier to remember from these right triangles:



Each triangle obeys Pythagoras. The squares of the legs add to the square of the hypotenuse. The first triangle has $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{a}$. Thus $x = a\sin\theta$ and $dx = a\cos\theta d\theta$.

Use these triangles in Problems 1–3 or use the table of substitutions in the text.

1. $\int_1^4 \frac{dx}{x^2\sqrt{x^2+9}}$ has a plus sign in the square root (*second triangle*).

   - Choose the second triangle with $a = 3$. Then $x = 3\tan\theta$ and $dx = 3\sec^2\theta d\theta$ and $\sqrt{x^2+9} = 3\sec\theta$. Substitute and then write $\sec\theta = \frac{1}{\cos\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$:

$$\int \frac{dx}{x^2\sqrt{x^2+9}} = \int \frac{3\sec^2\theta d\theta}{(9\tan^2\theta)(3\sec\theta)} = \int \frac{\cos\theta d\theta}{9\sin^2\theta} = \frac{-1}{9\sin\theta}.$$

   The integral was $\frac{1}{9}u^{-2}du$ with $u = \sin\theta$. The original limits of integration are $x = 1$ and $x = 4$. Instead of converting them to θ, we convert $\sin\theta$ back to x. The second triangle above shows

$$\frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{x^2+9}}{x} \text{ and then } \left[\frac{-\sqrt{x^2+9}}{9x}\right]_1^4 = \frac{-5}{36} + \frac{\sqrt{10}}{9} \approx 0.212.$$

2. $\int \sqrt{100 - x^2}\, dx$ contains the square root of $a^2 - x^2$ with $a = 10$.

   - Choose the *first* triangle: $x = 10\sin\theta$ and $\sqrt{100 - x^2} = 10\cos\theta$ and $dx = 10\cos\theta d\theta$:

$$\int \sqrt{100 - x^2}\, dx = \int (10\cos\theta) 10\cos\theta d\theta = 100\int \cos^2\theta d\theta = 100 \cdot \frac{1}{2}(\theta + \sin\theta\cos\theta) + C.$$

   Returning to x this is $50(\sin^{-1}\frac{x}{10} + \frac{x}{10}\cdot\frac{\sqrt{100-x^2}}{10}) = 50\sin^{-1}\frac{x}{10} + \frac{1}{2}x\sqrt{100 - x^2}$.

3. $\int \frac{dx}{x^2\sqrt{9x^2-25}}$ does not exactly contain $x^2 - a^2$. But try the third triangle.

   - Factor $\sqrt{9} = 3$ from the square root to leave $\sqrt{x^2 - \frac{25}{9}}$. Then $a^2 = \frac{25}{9}$ and $a = \frac{5}{3}$. The third triangle has $x = \frac{5}{3}\sec\theta$ and $dx = \frac{5}{3}\sec\theta\tan\theta d\theta$ and $\sqrt{9x^2 - 25} = 5\tan\theta$. The problem is now

$$\int \frac{\frac{5}{3}\sec\theta\tan\theta d\theta}{(\frac{5}{3})^2\sec^2\theta(5\tan\theta)} = \frac{3}{25}\int \cos\theta d\theta = \frac{3}{25}\sin\theta + C.$$

   The third triangle converts $\frac{3}{25}\sin\theta$ back to $\frac{\sqrt{9x^2-25}}{25x}$.

4. For $\int \frac{x^3 dx}{\sqrt{1-x^2}}$ the substitution $x = \sin\theta$ will work. But try $u = 1 - x^2$.

- Then $du = -2x\, dx$ and $x^2 = 1 - u$. The problem becomes $-\frac{1}{2}\int \frac{(1-u)du}{u^{1/2}} = \frac{1}{2}\int (u^{-1/2} - u^{-1/2})du$. In this case the old way is simpler than the new.

Problems 5 and 6 require *completing the square* before a trig substitution.

5. $\int \frac{dx}{\sqrt{x^2-8x+6}}$ requires us to complete $(x-4)^2$. We need $4^2 = 16$ so add and subtract 10 :

$$x^2 - 8x + 6 = (x^2 - 8x + 16) - 10 = (x-4)^2 - 10.$$

This has the form $u^2 - a^2$ with $u = x - 4$ and $a = \sqrt{10}$. Finally set $u = \sqrt{10}\sec\theta$:

$$\int \frac{dx}{\sqrt{x^2 - 8x + 6}} = \int \frac{du}{\sqrt{u^2 - 10}} = \int \frac{\sqrt{10}\sec\theta\tan\theta d\theta}{\sqrt{10}\tan\theta} = \int \sec\theta d\theta.$$

6. $\int \frac{\sqrt{2x-x^2}}{x}dx$ requires us to complete $2x - x^2$ (watch the minus sign):

$$2x - x^2 = -(x^2 - 2x) = -(x^2 - 2x + 1) + 1 = 1 - (x-1)^2.$$

This is $1 - u^2$ with $u = x - 1$ and $x = 1 + u$. The trig substitution is $u = \sin\theta$:

$$\int \frac{\sqrt{2x - x^2}}{x} = \int \frac{\sqrt{1-u^2}}{1+u}du = \int \frac{\cos^2\theta d\theta}{1+\sin\theta} = \int \frac{(1-\sin^2\theta)}{1+\sin\theta)}d\theta = \int (1-\sin\theta)d\theta.$$

Read-throughs and selected even-numbered solutions :

The function $\sqrt{1-x^2}$ suggests the substitution $x = \mathbf{\sin}\ \theta$. The square root becomes $\mathbf{\cos}\ \theta$ and dx changes to $\mathbf{\cos}\ \theta\ \mathbf{d\theta}$. The integral $\int(1-x^2)^{3/2}dx$ becomes $\int \mathbf{\cos^4}\theta\ d\theta$. The interval $\frac{1}{2} \le x \le 1$ changes to $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$.

For $\sqrt{a^2 - x^2}$ the substitution is $x = \mathbf{a\ \sin}\ \theta$ with $dx = \mathbf{a\ \cos}\ \theta\ \mathbf{d\theta}$. For $x^2 - a^2$ we use $x = \mathbf{a\ sec}\ \theta$ with $dx = \mathbf{a\ sec}\ \theta\ \mathbf{tan}\ \theta$. (**Insert:** For $x^2 + a^2$ use $x = a\ \tan\theta$). Then $\int dx/(1+x^2)$ becomes $\int\ d\theta$, because $1 + \tan^2\theta = \mathbf{sec^2}\theta$. The answer is $\theta = \tan^{-1} x$. We already knew that $\frac{\mathbf{1}}{\mathbf{1+x^2}}$ is the derivative of $\tan^{-1} x$.

The quadratic $x^2 + 2bx + c$ contains a **linear** term $2bx$. To remove it we **complete** the square. This gives $(x+b)^2 + C$ with $C = \mathbf{c - b^2}$. The example $x^2 + 4x + 9$ becomes $\mathbf{(x+2)^2 + 5}$. Then $u = x + 2$. In case x^2 enters with a minus sign, $-x^2 + 4x + 9$ becomes $\mathbf{-(x-2)^2 + 13}$. When the quadratic contains $4x^2$, start by factoring out **4**.

2 $x = a\sec\theta, x^2 - a^2 = a^2\tan^2\theta, \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} = \ln|\sec\theta + \tan\theta| = \ln|\frac{\mathbf{x}}{\mathbf{a}} + \sqrt{\frac{\mathbf{x^2}}{\mathbf{a^2}} - \mathbf{1}}| + C$

4 $x = \frac{1}{3}\tan\theta, 1 + 9x^2 = \sec^2\theta, \int \frac{dx}{1+9x^2} = \int \frac{\frac{1}{3}\sec^2\theta d\theta}{\sec^2\theta} = \frac{\theta}{3} = \mathbf{\frac{1}{3}\tan^{-1}3x} + C.$

12 Write $\sqrt{x^6 - x^8} = x^3\sqrt{1-x^2}$ and set $x = \sin\theta : \int \sqrt{x^6 - x^8}dx = \int \sin^3\theta\cos\theta(\cos\theta d\theta) =$
$\int \sin\theta(\cos^2\theta - \cos^4\theta)d\theta = -\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5} = \mathbf{-\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2}} + C$

14 $x = \sin\theta, \int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta d\theta}{\cos^3\theta} = \tan\theta + C = \frac{\mathbf{x}}{\mathbf{\sqrt{1-x^2}}} + C.$

32 First use geometry: $\int_{1/2}^{1}\sqrt{1-x^2}dx =$ half the area of the unit circle beyond $x = \frac{1}{2}$ which breaks into

$\frac{1}{2}(120°$ wedge minus $120°$ triangle$) = \frac{1}{2}(\frac{\pi}{3} - \frac{1}{2}\cdot\frac{1}{2}\cdot 2\sqrt{1-(\frac{1}{2})^2}) = \mathbf{\frac{\pi}{6} - \frac{\sqrt{3}}{8}}$.

Check by integration: $\int_{1/2}^{1}\sqrt{1-x^2}dx = [\frac{1}{2}(x\sqrt{1-x^2}+\sin^{-1}x)]_{1/2}^{1} = \frac{1}{2}(\frac{\pi}{2} - \frac{1}{2}\frac{\sqrt{3}}{2} - \frac{\pi}{6}) = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$.

34 $\int \frac{dx}{\cos x} = \int \sec x\, dx = \mathbf{\ln|\sec x + \tan x|} + C$; $\int \frac{dx}{1+\cos x}(\frac{1-\cos x}{1-\cos x}) = \int \frac{dx}{\sin^2 x} - \int \frac{\cos x\, dx}{\sin^2 x} = \int \csc^2 x\, dx - \int \frac{du}{u^2} = -\cot x + \frac{1}{\sin x} = \mathbf{\frac{1-\cos x}{\sin x}} + C$; $\int \frac{dx}{\sqrt{1+\cos x}} = \int \frac{dx}{\sqrt{2}\cos\frac{x}{2}} = \mathbf{\sqrt{2}\ln|\sec\frac{x}{2} + \tan\frac{x}{2}|} + C$

40 $x = \cosh\theta: \int \frac{\sqrt{x^2-1}}{x^2}dx = \int \frac{\sinh\theta}{\cosh^2\theta}\sinh\theta d\theta = \int \tanh^2\theta d\theta = \int(1-\text{sech}^2\theta)d\theta = \theta - \tanh\theta = \mathbf{\cosh^{-1}x - \frac{\sqrt{x^2-1}}{x}} + C$

44 $-x^2 + 2x + 8 = -(x-1)^2 + 9$

50 $\int \frac{dx}{\sqrt{9-(x-1)^2}} = \int \frac{du}{\sqrt{9-u^2}}$. Set $u = 3\sin\theta: \int \frac{\cos\theta d\theta}{\cos\theta} = \theta = \sin^{-1}\frac{u}{3} = \mathbf{\sin^{-1}\frac{x-1}{3}} + C$;

$\int \frac{dx}{10-x^2} = \mathbf{\frac{1}{2\sqrt{10}}\ln\frac{x-\sqrt{10}}{x+\sqrt{10}}} + C$; $\int \frac{dx}{(x+2)^2-16} = \int \frac{du}{u^2-16} = \frac{1}{8}\ln\frac{2u-8}{2u+8} = \mathbf{\frac{1}{8}\ln\frac{x-2}{x+6}} + C$

52 (a) $u = x-2$ (b) $u = x+1$ (c) $u = x-5$ (d) $u = x - \frac{1}{4}$

7.4 Partial Fractions (page 304)

This method applies to ratios $\frac{P(x)}{Q(x)}$, where P and Q are polynomials. The goal is to split the ratio into pieces that are easier to integrate. We begin by comparing this method with substitutions , on some basic problems where both methods give the answer.

1. $\int \frac{2x}{x^2-1}dx$. The substitution $u = x^2 - 1$ produces $\int \frac{du}{u} = \ln|u| = \ln|x^2-1|$.

   - Partial fractions breaks up this problem into smaller pieces:

   $$\frac{2x}{x^2-1} = \frac{2x}{(x+1)(x-1)} \text{ splits into } \frac{A}{x+1} + \frac{B}{x-1} = \frac{1}{x+1} + \frac{1}{x-1}.$$

   Now integrate the pieces to get $\ln|x+1| + \ln|x-1|$. This equals $\ln|x^2-1|$, the answer from substitution. We review how to find the numbers A and B starting from $\frac{2x}{x^2-1}$.

   - First, factor x^2-1 to get the denominators $x+1$ and $x-1$. Second, cover up $(x-1)$ and set $x = 1$:

   $$\frac{2x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \text{ becomes } \frac{2x}{(x+1)} = \frac{2}{2} = B. \text{ Thus } B = 1.$$

   Third, cover up $(x+1)$ and set $x = -1$ to find A:

   $$\text{At } x = -1 \text{ we get } A = \frac{2x}{(x-1)} = \frac{-2}{-2} = 1.$$

   That is it. Both methods are good. Use substitution or partial fractions.

2. $\int \frac{1}{x^2-1}dx$. The substitution $x = \sec\theta$ gives $\int \frac{4\sec\theta\tan\theta}{\tan^2\theta}d\theta = \int \frac{4}{\sin\theta}d\theta$.

   - The integral of $\frac{1}{\sin\theta}$ is not good. This time partial fractions look better:

   $$\frac{4}{x^2-1} = \frac{4}{(x+1)(x-1)} \text{ splits into } \frac{A}{x+1} + \frac{B}{x-1} = \frac{-2}{x+1} + \frac{2}{x-1}.$$

   The integral is $-2\ln|x+1| + 2\ln|x-1| = 2\ln|\frac{x-1}{x+1}|$. Remember the cover-up:

$$x = 1 \text{ gives } B = \frac{4}{(x+1)} = 2. \qquad x = -1 \text{ gives } A = \frac{4}{(x-1)} = -2.$$

3. $\int \frac{2x+4}{x^2-1}dx$ is the sum of the previous two integrals. Add A's and B's:

$$\frac{2x+4}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{-1}{x+1} + \frac{3}{x-1}.$$

In practice I would find $A = -1$ and $B = 3$ by the usual cover-up:

$$x = 1 \text{ gives } B = \frac{2x+4}{(x+1)} = \frac{6}{2}. \qquad x = -1 \text{ gives } A = \frac{2x+4}{(x-1)} = \frac{2}{-2}.$$

The integral is immediately $-\ln|x+1| + 3\ln|x-1|$. In this problem partial fractions is much better than substitutions. This case is $\frac{\text{linear}}{\text{quadratic}} = \frac{\text{degree 1}}{\text{degree 2}}$. *That is where partial fractions work best.*
The text solves the logistic equation by partial fractions. Here are more difficult ratios $\frac{P(x)}{Q(x)}$.

- It is the algebra, not the calculus, that can make $\frac{P(x)}{Q(x)}$ difficult. A reminder about division of polynomials may be helpful. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, *you first divide Q into P.* The example $\frac{x^3}{x^2+2x+1}$ requires long division:

$$\begin{array}{rll} & x & \text{divide } x^2 \text{ into } x^3 \text{ to get } x \\ x^2+2x+1 & \sqrt{x^3} & \\ & x^3+2x^2+x & \text{multiply } x^2+2x+1 \text{ by } x \\ \hline & -2x^2-x & \text{subtract from } x^3 \end{array}$$

The first part of the division gives x. If we stop there, division leaves $\frac{x^3}{x^2+2x+1} = x + \frac{-2x^2-x}{x^2+2x+1}$. This new fraction is $\frac{\text{degree 2}}{\text{degree 2}}$. So the division has to continue one more step:

$$\begin{array}{rll} & x-2 & \text{divide } x^2 \text{ into } -2x^2 \text{ to get } -2 \\ x^2+2x+1 & \sqrt{x^3} & \\ & x^3+2x^2+x & \\ \hline & -2x^2-x & \\ & -2x^2-4x-2 & \text{multiply } x^2+2x+1 \text{ by } -2 \\ \hline & 3x+2 & \text{subtract to find remainder} \end{array}$$

Now stop. The remainder $3x+2$ has lower degree than x^2+2x+1:

$$\frac{x^3}{x^2+2x+1} = x - 2 + \frac{3x+2}{x^2+2x+1} \text{ is ready for partial fractions.}$$

Factor x^2+2x+1 into $(x+1)^2$. Since $x+1$ is repeated, we look for

$$\frac{3x+2}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \quad \text{(notice this form!)}$$

Multiply through by $(x+1)^2$ to get $3x+2 = A(x+1)+B$. Set $x=-1$ to get $B=-1$. Set $x=0$ to get $A+B=2$. This makes $A=3$. The algebra is done and we integrate:

$$\int \frac{x^3}{x^2+2x+1}dx = \int (x-2+\frac{3x+2}{x^2+2x+1})dx = \int (x-2+\frac{3}{x+1} - \frac{1}{(x+1)^2})dx$$
$$= \frac{1}{2}x^2 - 2x + 3\ln|x+1| + (x+1)^{-1} + C.$$

4. $\int \frac{x^2+x}{x^2-4}dx$ also needs long division. The top and bottom have equal degree 2:

$$\begin{array}{rll} & 1 & \text{divide } x^2 \text{ into } x^2 \text{ to get } 1 \\ x^2+0x-4 & \overline{)\,x^2+x} & \\ & x^2+0x-4 & \text{multiply } x^2-4 \text{ by } 1 \\ \hline & x+4 & \text{subtract to find remainder } x+4 \end{array}$$

This says that $\frac{x^2+x}{x^2-4} = 1 + \frac{x+4}{x^2-4} = 1 + \frac{x+4}{(x-2)(x+2)}$. To decompose the remaining fraction, let

$$\frac{x+4}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Multiply by $x-2$ so the problem is $\frac{x+4}{x+2} = A + \frac{B(x-2)}{x+2}$. Set $x=2$ to get $A = \frac{6}{4} = \frac{3}{2}$. Cover up $x+2$ to get (in the mind's eye) $\frac{x+4}{x-2} = \frac{A(x+2)}{(x-2)} + B$. Set $x=-2$ to get $B = -\frac{1}{2}$. All together we have

$$\int \frac{x^2+x}{x^2-4}dx = \int (1 + \frac{\frac{3}{2}}{x-2} - \frac{\frac{1}{2}}{x+2})dx = x + \frac{3}{2}\ln|x-2| - \frac{1}{2}\ln|x+2| + C.$$

5. $\int \frac{7x^2+14x+15}{(x^2+3)(x+7)}dx$ requires no division. Why not? *We have degree* 2 *over degree* 3. Also x^2+3 cannot be factored further, so there are just two partial fractions:

$$\frac{7x^2+14x+15}{(x^2+3)(x+7)} = \frac{A}{x+7} + \frac{Bx+C}{x^2+3}. \qquad \text{Use } Bx+C \text{ over a quadratic, not just } B\,!$$

Cover up $x+7$ and set $x=-7$ to get $\frac{260}{52} = A$, or $A=5$. So far we have

$$\frac{7x^2+14x+15}{(x^2+3)(x+7)} = \frac{5}{x+7} + \frac{Bx+C}{x^2+3}.$$

We can set $x=0$ (because zero is easy) to get $\frac{15}{21} = \frac{5}{7} + \frac{C}{3}$, or $C=0$. Then set $x=-1$ to get $\frac{8}{24} = \frac{5}{6} + \frac{-B}{4}$. Thus $B=2$. Our integration problem is $\int (\frac{5}{x+7} + \frac{2x}{x^2+3})dx = 5\ln|x+7| + \ln(x^2+3) + C$.

6. (Problem 7.5.25) By substitution change $\int \frac{1+e^x}{1-e^x}$ to $\int \frac{P(u)}{Q(u)}du$. Then integrate.

- The ratio $\frac{1+e^x}{1-e^x}dx$ does not contain polynomials. Substitute $u=e^x$, $du = e^x\,dx$, and $dx = \frac{du}{u}$ to get $\frac{(1+u)du}{(1-u)u}$. A perfect set-up for partial fractions!

$$\frac{1+u}{u(1-u)} = \frac{A}{u} + \frac{B}{1-u} = \frac{1}{u} + \frac{2}{1-u}.$$

The integral is $\ln\, u - 2\ln|1-u| = x - 2\ln|1-e^x| + C$.

Read-throughs and selected even-numbered solutions :

The idea of **partial** fractions is to express $P(x)/Q(x)$ as a **sum** of simpler terms, each one easy to integrate. To begin, the degree of P should be **less than** the degree of Q. Then Q is split into **linear** factors like $x-5$ (possibly repeated) and quadratic factors like x^2+x+1 (possibly repeated). The quadratic factors have two **complex** roots, and do not allow real linear factors.

A factor like $x-5$ contributes a fraction $A/(\mathbf{x-5})$. Its integral is $\mathbf{A\ ln(x-5)}$. To compute A, cover up $\mathbf{x-5}$ in the denominator of P/Q. Then set $x = \mathbf{5}$, and the rest of P/Q becomes A. An equivalent method puts all fractions over a common denominator (which is **Q**). Then match the **numerators.** At the same point ($x = 5$) this matching gives A.

A repeated linear factor $(x-5)^2$ contributes not only $A/(x-5)$ but also $B/(\mathbf{x}-\mathbf{5})^{\mathbf{2}}$. A quadratic factor like x^2+x+1 contributes a fraction $(\mathbf{Cx}+\mathbf{D})/(x^2+x+1)$ involving C and D. A repeated quadratic factor or a triple linear factor would bring in $(Ex+F)/(x^2+x+1)^2$ or $G/(x-5)^3$. The conclusion is that any P/Q can be split into partial **fractions**, which can always be integrated.

6 $\frac{1}{x(x-1)(x+1)} = -\frac{1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$

14 $x+1\sqrt{x^2+0x+1}$ $\quad \frac{x^2+1}{x+1} = x-1+\frac{2}{x+1}$ **16** $\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}$

18 $\frac{x^2}{(x-3)(x+3)} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$ is impossible (no x^2 in the numerator on the right side).
Divide first to rewrite $\frac{x^2}{(x-3)(x+3)} = 1 + \frac{9}{(x-3)(x+3)} =$ (now use partial fractions) $1 + \frac{3/2}{x-3} - \frac{3/2}{x+3}$.

22 Set $u=\sqrt{x}$ so $u^2 = x$ and $2u\,du = dx$. Then $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}dx = \int \frac{1-u}{1+u}2u\,du =$ (*divide* $u+1$ *into* $-2u^2+2u$) $=$ $\int(-2u+4-\frac{4}{u+1})du = -u^2+4u-4\ln(u+1)+C = \mathbf{-x+4\sqrt{x}-4\ln(\sqrt{x}+1)} + C$.

7.5 Improper Integrals (page 309)

An improper integral is really a limit: $\int_0^\infty y(x)dx$ means $\lim_{b\to\infty}\int_0^b y(x)dx$. Usually we just integrate and substitute $b=\infty$. If the integral of $y(x)$ contains e^{-x} then $e^{-\infty}=0$. If the integral contains $\frac{1}{x}$ or $\frac{1}{x+7}$ then $\frac{1}{\infty}=0$. If the integral contains $\tan^{-1}x$ then $\tan^{-1}\infty = \frac{\pi}{2}$. The numbers are often convenient when the upper limit is $b=\infty$.

Similarly $\int_{-\infty}^{\infty} y(x)dx$ is really the sum of two limits. You have to use a and b to keep those limits separate: $\lim_{a\to-\infty}\int_a^0 y(x)dx + \lim_{b\to\infty}\int_0^b y(x)dx$. Normally just integrate $y(x)$ and substitute $a=-\infty$ and $b=\infty$.

EXAMPLE 1 $\int_{-\infty}^{\infty}\frac{dx}{x^2+5} = [\frac{1}{5}\tan^{-1}\frac{x}{5}]_{-\infty}^{\infty} = \frac{1}{5}(\frac{\pi}{2}) - \frac{1}{5}(-\frac{\pi}{2}) = \frac{\pi}{5}$.

Notice the lower limit, where $\tan^{-1}\frac{a}{5}$ approaches $-\frac{\pi}{2}$ as a approaches $-\infty$. Strictly speaking the solution should have separated the limits $a=-\infty$ and $b=\infty$:

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+5} = \lim_{a\to-\infty}[\frac{1}{5}\tan^{-1}\frac{x}{5}]_a^0 + \lim_{b\to\infty}[\frac{1}{5}\tan^{-1}\frac{x}{5}]_0^b.$$

If $y(x)$ blows up *inside* the interval, the integral is really the sum of a left-hand limit and a right-hand limit.

EXAMPLE 2 $\int_{-2}^{3}\frac{dx}{x^3}$ blows up at $x=0$ inside the interval.

- If this was not in a section labeled "improper integrals," would your answer have been $[-\frac{1}{2}x^{-2}]_{-2}^{3} = \frac{1}{2}(\frac{1}{9}-\frac{1}{4}) = \frac{5}{72}$? This is a very easy mistake to make. But since $\frac{1}{x^3}$ is infinite at $x=0$, the integral is improper. Separate it into the part up to $x=0$ and the part beyond $x=0$.

  The integral of $\frac{1}{x^3}$ is $\frac{-1}{2x^2}$ which blows up at $x=0$. Those integrals from -2 to 0 and from 0 to 3 are both infinite. This improper integral *diverges*.

- Notice: The question is whether the *integral* blows up, not whether $y(x)$ blows up. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is OK.

  Lots of times you only need to know whether or not the integral converges. This is where the comparison test comes in. Assuming $y(x)$ is positive, try to show that its unknown integral is smaller than a known finite integral (or greater than a known divergent integral).

EXAMPLE 3 $\int_1^\infty \frac{2+\cos x}{x^3}dx$ has $2+\cos x$ between 1 and 3. Therefore $\frac{2+\cos x}{x^3} \le \frac{3}{x^3}$. Since $\int_1^\infty \frac{3}{x^3}dx$ converges to a finite answer, the original integral must converge. You could have started with $\frac{2+\cos x}{x^3} \ge \frac{1}{x^3}$. This is true, but it is not helpful! It only shows that the integral is greater than a convergent integral. The greater one could converge or diverge – this comparison doesn't tell.

EXAMPLE 4 $\int_5^\infty \frac{\sqrt{x}}{x+8}dx$ has $\frac{\sqrt{x}}{x+8} \approx \frac{\sqrt{x}}{x} = \frac{1}{x^{1/2}}$ for large x.
We suspect divergence (the area under $x^{-1/2}$ is infinite). To show a comparison, note $\frac{\sqrt{x}}{x+8} > \frac{\sqrt{x}}{3x}$. This is because 8 is smaller than $2x$ beyond our lower limit $x = 5$. Increasing the denominator to $3x$ makes the fraction smaller. The official reasoning is

$$\int_5^\infty \frac{\sqrt{x}\,dx}{x+8} > \int_5^\infty \frac{\sqrt{x}}{3x}dx = \int_5^\infty \frac{dx}{3x^{1/2}} = \lim_{b\to\infty} \frac{2}{3}x^{1/2}]_5^b = \infty.$$

5. (Problem 7.5.37) What is improper about the area between $y = \sec x$ and $y = \tan x$?

   The area under the secant graph minus the area under the tangent graph is

$$\int_0^{\pi/2} \sec x\ dx - \int_0^{\pi/2} \tan x\ dx = \ln(\sec x + \tan x)]_0^{\pi/2} + \ln(\cos x)]_0^{\pi/2} = \infty - \infty.$$

   The separate areas are infinite! However we can subtract before integrating:

$$\begin{aligned}\int_0^{\pi/2}(\sec x - \tan x)dx &= [\ln(\sec x + \tan x) + \ln(\cos x)]_0^{\pi/2}\\ &= [\ln(\cos x)(\sec x + \tan x)]_0^{\pi/2} = [\ln(1+\sin x)]_0^{\pi/2} = \ln 2 - \ln 1 = \ln 2.\end{aligned}$$

   This is perfectly correct. The difference of areas comes in Section 8.1.

Read-throughs and selected even-numbered solutions :

An improper integral $\int_a^b y(x)dx$ has lower limit $a = -\infty$ or upper limit $b = \infty$ or y becomes **infinite** in the interval $a \le x \le b$. The example $\int_1^\infty dx/x^3$ is improper because $\mathbf{b = \infty}$. We should study the limit of $\int_1^b dx/x^3$ as $\mathbf{b \to \infty}$. In practice we work directly with $-\frac{1}{2}x^{-2}]_1^\infty = \mathbf{\frac{1}{2}}$. For $p > 1$ the improper integral $\int_p^\infty x^{-p}dx$ is finite. For $p < 1$ the improper integral $\int_0^1 x^{-p}dx$ is finite. For $y = e^{-x}$ the integral from 0 to ∞ is **1**.

Suppose $0 \le u(x) \le v(x)$ for all x. The convergence of $\int \mathbf{v(x)\ dx}$ implies the convergence of $\int \mathbf{u(x)dx}$. The divergence of $\int u(x)dx$ **implies** the divergence of $\int v(x)dx$. From $-\infty$ to ∞, the integral of $1/(e^x + e^{-x})$ converges by comparison with $\mathbf{1/e^{|x|}}$. Strictly speaking we split $(-\infty,\infty)$ into $(-\infty, 0)$ and $(0,\infty)$. Changing to $1/(e^x - e^{-x})$ gives divergence, because $\mathbf{e^x = e^{-x}}$ **at** $\mathbf{x = 0}$. Also $\int_{-\pi}^{\pi} dx/\sin x$ diverges by comparison with $\int \mathbf{dx/x}$. The regions left and right of zero don't cancel because $\infty - \infty$ is **not zero**.

2 $\int_0^1 \frac{dx}{x^\pi} = [\frac{x^{1-\pi}}{1-\pi}]_0^1$ diverges at $x = 0$: infinite area
8 $\int_{-\infty}^{\infty} \sin x\, dx$ is not defined because $\int_a^b \sin x\, dx = \cos a - \cos b$ does not approach a limit as $b \to \infty$ and $a \to -\infty$
16 $\int_0^\infty \frac{e^x\, dx}{(e^x-1)^p} =$ (set $u = e^x - 1$) $\int_0^\infty \frac{du}{u^p}$ which is infinite: diverges at $u = 0$ if $p \geq 1$, diverges at $u = \infty$ if $p \leq 1$.
18 $\int_0^1 \frac{dx}{x^6+1} < \int_0^1 \frac{dx}{1} = 1$: convergence
24 $\int_0^1 \sqrt{-\ln x}dx < \int_0^{1/e}(-\ln x)dx + \int_{1/e}^1 1dx = [-x\ln x + x]_0^{1/e} + [x]_{1/e}^1 = \frac{1}{e} + 1$: convergence (note $x \ln x \to 0$ as $x \to 0$)
36 $\int_a^b \frac{x\, dx}{1+x^2} = [\frac{1}{2}\ln(1+x^2)]_a^b = \frac{1}{2}\ln(1+b^2) - \frac{1}{2}\ln(1+a^2)$. As $b \to \infty$ or as $a \to -\infty$ (separately!) there is no limiting value. If $a = -b$ then the answer is zero – but we are not allowed to connect a and b.
40 The red area in the right figure has an extra unit square (area 1) compared to the red area on the left.

7 Chapter Review Problems

Review Problems

R1 Why is $\int u(x)v(x)dx$ not equal to $(\int u(x)dx)(\int v(x)dx)$? What formula is correct?

R2 What method of integration would you use for these integrals?

$\int x\cos(2x^2+1)dx \quad \int x\cos(2x+1) \quad \int \cos^2(2x+1)dx \quad \int \cos(2x+1)\sin(2x+1)dx$

$\int \cos^3(2x+1)\sin^5(2x+1)dx \quad \int \cos^4(2x+1)\sin^2(2x+1)dx \quad \int \cos 2x \sin 3x\, dx \quad \int \frac{\cos(2x+1)}{\sin(2x+1)}dx$

R3 Which eight methods will succeed for these eight integrals?

$\int \frac{x}{\sqrt{3+x^2}}dx \quad \int \frac{dx}{\sqrt{3+x^2}} \quad \int \frac{dx}{\sqrt{3-x^2}} \quad \int \frac{dx}{x^2-3}$

$\int \frac{dx}{x^2+2x-3} \quad \int \frac{dx}{\sqrt{x^2+2x}} \quad \int \frac{x\, dx}{x^2-3} \quad \int \frac{x^3 dx}{x^2-3}$

R4 What is an improper integral? Show by example four ways a definite integral can be improper.

R5 Explain with two pictures the comparison tests for convergence and divergence of improper integrals.

Drill Problems

D1 $\int x^2 \ln x\, dx$

D2 $\int e^x \sin 2x\, dx$

D3 $\int x^3\sqrt{1-x^2}\, dx$

D4 $\int \frac{x^3\, dx}{x^2+4x+3}$

D5 $\int \frac{\ln x}{\sqrt{x}}\, dx$

D6 $\int \tan^3 2x\ \sec^2 2x\, dx$

D7 $\int e^{e^x} e^x\, dx$

D8 $\int \frac{dx}{\sqrt{3-2x-x^2}}$

D9 $\int \sin(\ln x)\, dx$

D10 $\int \frac{6x+4}{x^3-4x}\, dx$

D11 $\int \sin^{-1}\sqrt{x}\, dx$

D12 $\int \cos^4 2x \sin^2 2x\, dx$

D13 $\int \frac{dx}{(4-x^2)^{3/2}}$

D14 $\int \frac{9x+36}{x^3+6x^2+9x}\, dx$

Evaluate the improper integrals **D15** to **D20** or show that they diverge.

D15 $\int_0^{\pi/2} \frac{\cos x\, dx}{1-\sin x}$

D16 $\int_1^{\infty} \frac{\ln x}{\sqrt{x}} dx$

D17 $\int_0^{\infty} xe^{-x} dx$

D18 $\int_{-1}^{\infty} x^{-1/3} dx$

D19 $\int_0^{33} \frac{dx}{(1-x)^{2/5}}$

D20 $\int_e^{\infty} \frac{dx}{x(\ln x)^{3/2}}$