CHAPTER 9 POLAR COORDINATES AND COMPLEX NUMBERS

9.1 Polar Coordinates (page 350)

Polar coordinates r and θ correspond to $x = \mathbf{r \cos \theta}$ and $y = \mathbf{r \sin \theta}$. The points with $r > 0$ and $\theta = \pi$ are located **on the negative x axis**. The points with $r = 1$ and $0 \le \theta \le \pi$ are located on **a semicircle**. Reversing the sign of θ moves the point (x, y) to $(\mathbf{x}, -\mathbf{y})$.

Given x and y, the polar distance is $r = \sqrt{\mathbf{x^2+y^2}}$. The tangent of θ is $\mathbf{y/x}$. The point (6,8) has $r = \mathbf{10}$ and $\theta = \mathbf{\tan^{-1}\frac{8}{6}}$. Another point with the same θ is $\mathbf{(3,4)}$. Another point with the same r is **(10,0)**. Another point with the same r and $\tan\theta$ is $\mathbf{(-6,-8)}$.

The polar equation $r = \cos\theta$ produces a shifted **circle**. The top point is at $\theta = \pi/4$, which gives $r = \sqrt{2}/2$. When θ goes from 0 to 2π, we go **two** times around the graph. Rewriting as $r^2 = r\cos\theta$ leads to the xy equation $\mathbf{x^2+y^2 = x}$. Substituting $r = \cos\theta$ into $x = r\cos\theta$ yields $x = \mathbf{\cos^2\theta}$ and similarly $y = \mathbf{\cos\theta \sin\theta}$. In this form x and y are functions of the **parameter** θ.

1 $(1, \frac{\pi}{2})$ **3** $(2, \frac{\pi}{4})$ **5** $(\sqrt{2}, \frac{5\pi}{4})$ **7** $(0,2)$ **9** $(\sqrt{10}, \sqrt{10})$ **11** $(\sqrt{3}, -1)$ **13** $2\sqrt{2}$
15 $\sqrt{r^2+R^2-2rR\cos(\theta-\phi)}$
17 $0<r<\infty, -\frac{\pi}{2}<\theta<\frac{\pi}{2}; 0\le r<\infty, \pi\le\theta\le 2\pi; \sqrt{4}<r<\sqrt{5}, 0\le\theta<2\pi; 0\le r<\infty, -\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$
19 $y = x\tan\theta, r = x\sec\theta$ **21** $\theta = \frac{\pi}{4}$, all r; $r = \frac{1}{\sin\theta+\cos\theta}$; $r = \cos\theta+\sin\theta$
23 $x^2+y^2 = y$ **25** $x = r\sin\theta\cos\theta, y = r\sin^2\theta, x^2+y^2 = y$
27 $x^2+y^2 = x+y, (x-\frac{1}{2})^2+(y-\frac{1}{2})^2 = (\frac{\sqrt{2}}{2})^2$ **29** $x = \frac{\cos\theta}{\cos\theta+\sin\theta}, y = \frac{\sin\theta}{\cos\theta+\sin\theta}$ **31** $(x^2+y^2)^3 = x^4$

2 $x=-4, y=0$ has polar coordinates $\mathbf{r=4, \theta=\pi}$
4 $x=-1, y=\sqrt{3}$ has polar coordinates $\mathbf{r = 2}, \theta = \frac{2\pi}{3}$.
6 $x=3, y=4$ has polar coordinates $\mathbf{r=5}, \theta = \mathbf{\tan^{-1}(\frac{4}{3})} = .925$.
8 $r=1, \theta=\frac{3\pi}{2}$ has rectangular coordinates $\mathbf{x=0, y=-1}$.
10 $r=3\pi, \theta=3\pi$ has rectangular coordinates $\mathbf{x=-3\pi, y=0}$
12 $r=2, \theta=\frac{5\pi}{6}$ has rectangular coordinates $\mathbf{x=-\sqrt{3}, y=1}$
14 The distance is 5. Better question with same answer: how far is $(3, \frac{\pi}{3})$ from $(4, \frac{2\pi}{3})$?
16 (a) $(-1, \frac{\pi}{2})$ is the same point as $(1, \frac{3\pi}{2})$ or $(-1, \frac{5\pi}{2})$ or $\cdots$ (b) $(-1, \frac{3\pi}{4})$ is the same point as $(1, \frac{7\pi}{4})$ or $(-1, -\frac{\pi}{4})$ or $\cdots$ (c) $(1, -\frac{\pi}{2})$ is the same point as $(-1, \frac{\pi}{2})$ or $(1, \frac{3\pi}{2})$ or $\cdots$ (d) $r=0, \theta=0$ is the same point as $\mathbf{r=0}, \theta =$ **any angle**.
18 (a) **False** ($r=1, \theta=\frac{\pi}{4}$ is a different point from $r=-1, \theta=-\frac{\pi}{4}$) (b) **False** (for fixed r we can add any multiple of 2π to θ) (c) **True** ($r\sin\theta = 1$ is the horizontal line $y=1$).
20 $x=\sqrt{3}, y=1$ yields $r=2, \tan\theta = \frac{1}{\sqrt{3}}$. So does $\mathbf{x=-\sqrt{3}, y=-1}$.
22 Take the line from (0,0) to (r_1, θ_1) as the base (its length is r_1). The height of the third point (r_2, θ_2), measured perpendicular to this base, is r_2 times $\sin(\theta_2-\theta_1)$.
24 The 13 values $\theta = 0°, 30°, \cdots, 360°$ give six different points with $r = \sin\theta$. To go once around the circle take $0 \le \theta < \pi$.

26 From $x = \cos^2\theta$ and $y = \sin\theta\cos\theta$, square and add to find $\mathbf{x^2 + y^2} = \cos^2\theta(\cos^2\theta + \sin^2\theta) = \cos^2\theta = \mathbf{x}$.

28 Multiply $r = a\cos\theta + b\sin\theta$ by r to find $x^2 + y^2 = ax + by$. Complete squares in $x^2 - ax = (x - \frac{a}{2})^2 - (\frac{a}{2})^2$ and similarly in $y^2 - by$ to find $(x - \frac{a}{2})^2 + (y - \frac{b}{2})^2 = (\frac{a}{2})^2 + (\frac{b}{2})^2$. This is a circle centered at $(\frac{a}{2}, \frac{b}{2})$ with radius $r = \sqrt{(\frac{a}{2})^2 + (\frac{b}{2})^2} = \mathbf{\frac{1}{2}\sqrt{a^2 + b^2}}$.

30 The point $x = \cos^2\theta, y = \sin^2\theta$ is generally not at the polar angle θ. For example let $\theta = \frac{\pi}{6} = 30°$: then $x = \frac{3}{4}$ and $y = \frac{1}{4}$. The polar angle for this point has tangent $= \frac{1}{3}$, but the tangent of $\frac{\pi}{6}$ is $\frac{1}{\sqrt{3}}$. Conclusion: an angle named θ is not automatically the polar angle. **See Problem 9.3.40.**

32 The second figure is not a closed curve as it stands. As the parameter t keeps going, the spaces around the circle fill up and the curve eventually closes (but the figure becomes less beautiful).

9.2 Polar Equations and Graphs (page 355)

The circle of radius 3 around the origin has polar equation $\mathbf{r = 3}$. The 45° line has polar equation $\theta = \pi/4$. Those graphs meet at an angle of **90°**. Multiplying $r = 4\cos\theta$ by r yields the xy equation $\mathbf{x^2 + y^2 = 4x}$. Its graph is a **circle** with center at **(2,0)**. The graph of $r = 4/\cos\theta$ is the line $x = \mathbf{4}$. The equation $r^2 = \cos 2\theta$ is not changed when $\theta \to -\theta$ (symmetric across **the x axis**) and when $\theta \to \pi + \theta$ (or $r \to -\mathbf{r}$). The graph of $r = 1 + \cos\theta$ is a **cardioid**.

The graph of $r = A/(1 + \mathbf{e}\cos\theta)$ is a conic section with one focus at $(\mathbf{0, 0})$. It is an ellipse if $\mathbf{e < 1}$ and a hyperbola if $\mathbf{e > 1}$. The equation $r = 1/(1 + \cos\theta)$ leads to $r + x = 1$ which gives a **parabola**. Then r = distance from origin equals $1 - x$ = distance from **directrix** $\mathbf{y = 1}$. The equations $r = 3(1 - x)$ and $r = \frac{1}{3}(1 - x)$ represent a **hyperbola** and an **ellipse**. Including a shift and rotation, conics are determined by **five** numbers.

1 Line $y = 1$ **3** Circle $x^2 + y^2 = 2x$ **5** Ellipse $3x^2 + 4y^2 = 1 - 2x$ **7** x, y, r symmetries
9 x symmetry only **11** No symmetry **13** x, y, r symmetries!
15 $x^2 + y^2 = 6y + 8x \to (x - 4)^2 + (y - 3)^2 = 5^2$, center (4,3) **17** (2,0), (0,0)
19 $r = 1 - \frac{\sqrt{2}}{2}, \theta = \frac{3\pi}{4}; r = 1 + \frac{\sqrt{2}}{2}, \theta = \frac{7\pi}{4}; (0,0)$ **21** $r = 2, \theta = \pm\frac{\pi}{12}, \pm\frac{5\pi}{12}, \pm\frac{7\pi}{12}, \pm\frac{11\pi}{12}$
23 $(x, y) = (1, 1)$ **25** $r = \cos 5\theta$ has 5 petals **27** $(x^2 + y^2 - x)^2 = x^2 + y^2$
29 $(x^2 + y^2)^3 = (x^2 - y^2)^2$ **31** $\cos\theta = -\frac{1}{2}, \sin\theta = \frac{\sqrt{3}}{2} \to y = \frac{2\sqrt{3}}{3}, x = -\frac{2}{3}$ **33** $x = \frac{4}{3}, r = -\frac{5}{3}$ **35** .967

2 $r\cos\theta - r\sin\theta = 2$ is the straight line $\mathbf{x - y = 2}$.

4 $r = -2\sin\theta$ is the circle $r^2 = -2r\sin\theta$ or $x^2 + y^2 = -2y$ or $\mathbf{x^2 + (y + 1)^2 = 1}$; below the origin with center at $(0, -1)$ and radius 1.

6 $r = \frac{1}{1 + 2\cos\theta}$ is the hyperbola of Example 7 and Figure 9.5c: $r + 2r\cos\theta = 1$ is $r = 1 - 2x$ or $x^2 + y^2 = 1 - 4x + 4x^2$. The figure should show $r = -1$ and $\theta = \pi$ on the right branch.

8 $r^2 = 4\sin 2\theta$ has loops in the first and third quadrants. It possesses **r symmetry** (change r to $-r$ and the equation is unchanged). Changing θ to $-\theta$ or $\pi - \theta$ or $2\pi - \theta$ reverses the sign of the right hand side.

10 $r^2 = 10 + 6\cos 4\theta$ has x, y, and r symmetry. It comes in from $r = 4$ at $\theta = 0$ to $r = 2$ at $\theta = \frac{\pi}{4}$ and back out to $r = 4$ at $\theta = \frac{\pi}{2}$. Repeat in each quadrant to form a "star-fish".

12 $r = 1/\theta$ has **y symmetry**. Change θ to $-\theta$ and r to $-r$: same equation (θ to $\pi - \theta$ gives a different equation:

must try both tests.) Note that the maximum of $y = r\sin\theta = \frac{\sin\theta}{\theta}$ is $y = 1$ as $\theta \to 0$: the line $y = 1$ is a ***horizontal asymptote!*** As negative θ approach zero, the spiral goes left toward the same asymptote $y = 1$.

14 $r = 1 - 2\sin 3\theta$ has **y axis symmetry**: change θ to $\pi - \theta$, then $\sin 3(\pi - \theta) = \sin(\pi - 3\theta) = \sin 3\theta$.

16 This is another case where **the parameter t is not the polar angle.** (The Earth completed a circle at $t = 1$.)

18 $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$ are lemniscates (or "spectacles"). They meet when $\sin 2\theta = \cos 2\theta$ or $2\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$ or $\theta = \frac{\pi}{8}$ or $\frac{5\pi}{8}$ (r can be positive or negative). They also meet at the origin $\mathbf{r = 0}$.

20 $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$ are cardioids (reaching right to $r = 2, \theta = 0$ and left to $r = 2, \theta = \pi$). They meet when $\cos\theta = 0$ at $\mathbf{r = 1}, \theta = \frac{\pi}{2}$ and $\mathbf{r = 1}, \theta = \frac{3\pi}{2}$. They also meet at the origin $\mathbf{r = 0}$.

22 If $\cos\theta = \frac{r^2}{4}$ and $\cos\theta = 1 - r$ then $\frac{r^2}{4} = 1 - r$ and $r^2 + 4r - 4 = 0$. This gives $r = -2 - \sqrt{8}$ and $\mathbf{r = -2 + \sqrt{8}}$. The first r is negative and cannot equal $1 - \cos\theta$. The second gives $\cos\theta = 1 - r = 3 - \sqrt{8}$ and $\theta \approx \mathbf{80^\circ}$ **or** $\theta \approx \mathbf{-80^\circ}$. The curves also meet at **the origin** $\mathbf{r = 0}$ and at the point $\mathbf{r = -2}, \theta = \mathbf{0}$ which is also $\mathbf{r = +2}, \theta = \pi$.

24 The limacon $r = 1 + b\cos\theta$ has $x = r\cos\theta = \cos\theta + b\cos^2\theta$ and $\frac{dx}{d\theta} = -\sin\theta - 2b\cos\theta\sin\theta$. Then $\frac{d^2x}{d\theta^2} = -\cos\theta - 2b\cos^2\theta + 2b\sin^2\theta$ which equals $1 - 2b$ at $\theta = \pi$. **The dimple begins at** $\mathbf{b = \frac{1}{2}}$. At $b = 1$ it becomes the cusp in the cardioid.

26 The other 101 petals in $r = \cos 101\theta$ are **duplicates of the first 101.** For example $\theta = \pi$ gives $r = \cos 101\pi = -1$ which is also $\theta = 0, r = +1$. (Note that $\cos 100\pi = +1$ gives a new point.)

28 (a) Yes, x and y symmetry imply r symmetry. Reflections across the x axis and then the y axis take (x, y) to $(x, -y)$ to $(-x, -y)$ which is reflection through the origin. (b) The point $r = -1, \theta = \frac{3\pi}{2}$ satisfies the equation $r = \cos 2\theta$ and it is the same point as $r = 1, \theta = \frac{\pi}{2}$.

30 (a) $r^2 = \theta$ (b) $x + y = 1$ or $r\cos\theta + r\sin\theta = 1$ or $r = \frac{1}{\cos\theta + \sin\theta}$ (c) ellipse $x^2 + 2y^2 = 1$ or $r^2(\cos^2\theta + 2\sin^2\theta) = 1$

32 (a) $\theta = \frac{\pi}{2}$ gives $r = 1$; this is $x = 0, y = 1$ (b) The graph crosses the x axis at $\theta = 0$ and π where $x = \frac{1}{1+e}$ and $x = \frac{-1}{1-e}$. The center of the graph is halfway between at $x = \frac{1}{2}\left(\frac{1}{1+e} - \frac{1}{1-e}\right) = \frac{-e}{1-e^2}$. The second focus is twice as far from the origin at $\mathbf{\frac{-2e}{1-e^2}}$. (Check: $e = 0$ gives center of circle, $e = 1$ gives second focus of parabola at infinity.)

34 $r = \frac{A}{1+e\cos\theta}$ and $r = \frac{1}{C+D\cos\theta}$ are the same if $\mathbf{C = \frac{1}{A}}$ and $\mathbf{D = \frac{e}{A}}$. For the mirror image across the y axis, θ becomes $\pi - \theta$ and $\cos\theta$ changes sign.

36 Maximize $y = \frac{A\sin\theta}{1+e\cos\theta}$ where $\frac{dy}{d\theta} = \frac{(1+e\cos\theta)A\cos\theta + (A\sin\theta)e\sin\theta}{(1+e\cos\theta)^2} = 0$. Then $A\cos\theta + Ae = 0$ or $\cos\theta = -e$ and $y_{\max} = \frac{A\sqrt{1-e^2}}{1-e^2} = \frac{A}{\sqrt{1-e^2}}$ (which equals b in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$).

9.3 Slope, Length, and Area for Polar Curves (page 359)

A circular wedge with angle $\Delta\theta$ is a fraction $\mathbf{\Delta\theta/2\pi}$ of a whole circle. If the radius is r, the wedge area is $\mathbf{\frac{1}{2}r^2\Delta\theta}$. Then the area inside $r = F(\theta)$ is $\int \mathbf{\frac{1}{2}r^2 d\theta} = \int \mathbf{\frac{1}{2}(F(\theta))^2 d\theta}$. The area inside $r = \theta^2$ from 0 to π is $\mathbf{\pi^5/10}$. That spiral meets the circle $r = 1$ at $\theta = \mathbf{1}$. The area inside the circle and outside the spiral is $\mathbf{\frac{1}{2} - \frac{1}{10}}$. A chopped wedge of angle $\Delta\theta$ between r_1 and r_2 has area $\mathbf{\frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta}$.

The curve $r = F(\theta)$ has $x = r\cos\theta = \mathbf{F(\theta)\cos\theta}$ and $y = \mathbf{F(\theta)\sin\theta}$. The slope dy/dx is $dy/d\theta$ divided by $\mathbf{dx/d\theta}$. For length $(ds)^2 = (dx)^2 + (dy)^2 = \mathbf{(dr)^2 + (rd\theta)^2}$. The length of the spiral $r = \theta$ to $\theta = \pi$ is

$\int\sqrt{1+\theta^2}d\theta$. The surface area when $r=\theta$ is revolved around the x axis is $\int 2\pi y\,ds = \int 2\pi\theta\sin\theta\sqrt{1+\theta^2}d\theta$. The volume of that solid is $\int \pi y^2 dx = \int \pi\theta^2\sin^2\theta\,(\cos\theta - \theta\sin\theta)d\theta$.

1 Area $\frac{3\pi}{2}$ **3** Area $\frac{9\pi}{2}$ **5** Area $\frac{\pi}{8}$ **7** Area $\frac{\pi}{8}-\frac{1}{4}$ **9** $\int_{-\pi/3}^{\pi/3}(\frac{9}{2}\cos^2\theta - \frac{(1+\cos\theta)^2}{2})d\theta = \pi$

11 Area 8π **13** Only allow $r^2>0$, then $4\int_0^{\pi/4}\frac{1}{2}\cos 2\theta\,d\theta = 1$ **15** $2+\frac{\pi}{4}$

17 $\theta=0$; left points $r=\frac{1}{2}, \theta=\pm\frac{2\pi}{3}, x=-\frac{1}{4}, y=\pm\frac{\sqrt{3}}{4}$

19 $\frac{r^2}{2c}]_6^{14} = 40{,}000$; $\frac{1}{2c}[r\sqrt{r^2+c^2}+c^2\ln(r+\sqrt{r^2+c^2})]_6^{14} = 40{,}000.001$

21 $\tan\psi = \tan\theta$ **23** $x=0, y=1$ is on limacon but not circle **25** $\frac{1}{2}\ln(2\pi+\sqrt{1+4\pi^2})+\pi\sqrt{1+4\pi^2}$

27 $\frac{3\pi}{2}$ **29** $\frac{1}{2}$ (base)(height) $\approx \frac{1}{2}(r\Delta\theta)r$ **31** $\frac{4\pi}{5}\sqrt{2}$ **33** $2\pi(2-\sqrt{2})$ **35** $\frac{8\pi}{3}$ **39** $\sec\theta$

2 $A=\int\frac{1}{2}r^2d\theta = \int_0^\pi \frac{1}{2}(\sin\theta+\cos\theta)^2d\theta = \int_0^\pi\frac{1}{2}(\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta)d\theta = \int_0^\pi\frac{1}{2}(1+\sin 2\theta)d\theta$ $=[\frac{\theta}{2}-\frac{\cos 2\theta}{4}]_0^\pi = \frac{\pi}{2}$. This is the area of the circle $r^2 = x+y$ or $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{2}$

4 The inner loop is where $r<0$ or $\cos\theta<-\frac{1}{2}$ or $\frac{2\pi}{3}<\theta<\frac{4\pi}{3}$. Its area is $\int\frac{r^2}{2}d\theta = \int\frac{1}{2}(1+4\cos\theta+4\cos^2\theta)d\theta =$ $[\frac{\theta}{2}+2\sin\theta+\theta+\cos\theta\,\sin\theta]_{2\pi/3}^{4\pi/3} = \frac{\pi}{3}-2(\sqrt{3})+\frac{2\pi}{3}+\frac{1}{2}\sqrt{3} = \pi-\frac{3}{2}\sqrt{3}$.

6 A petal begins and ends at $r=0$. For $r=\cos 3\theta$ this is from $\theta=-\frac{\pi}{6}$ to $\frac{\pi}{6}$. The area is $\int\frac{1}{2}\cos^2 3\theta\,d\theta = \frac{1}{4}\int(1+\cos 6\theta)d\theta = [\frac{\theta}{4}+\frac{\sin 6\theta}{24}]_{-\pi/6}^{\pi/6} = \frac{\pi}{12}$.

8 The y axis is $\theta=\frac{\pi}{2}$. The area is $\int\frac{1}{2}r^2d\theta = \int_0^{\pi/2}\frac{1}{2}\theta^2d\theta = [\frac{\theta^3}{6}]_0^{\pi/2} = \frac{\pi^3}{48}$.

10 $r^2=4\cos 2\theta$ meets $r^2=2$ when $\cos 2\theta=\frac{1}{2}$ or $2\theta=\pm\frac{\pi}{3}$ and $\pm(\frac{\pi}{3}+2\pi)$. Then $\theta=\pm\frac{\pi}{6}$ and $\pm\frac{5\pi}{6}$. By symmetry, integrate from 0 to $\frac{\pi}{6}$ and multiply by 4. Area $=4\int_0^{\pi/6}\frac{1}{2}(r_1^2-r_2^2)d\theta = 2\int_0^{\pi/6}(4\cos 2\theta-2)d\theta =$ $[4\sin 2\theta - 4\theta]_0^{\pi/6} = 4(\frac{\sqrt{3}}{2})-4(\frac{\pi}{6}) = 2\sqrt{3}-\frac{2\pi}{3}$.

12 Intersection when $10\cos\theta = 6$ or $\cos\theta=.6$. Area $\int\frac{1}{2}(r_1^2-r_2^2)d\theta = \int_0^{\cos^{-1}.6}(100-\frac{36}{\cos^2\theta})d\theta =$ $[100\theta-36\tan\theta]_0^{\cos^{-1}.6} = 100\cos^{-1}.6 - 36(\frac{4}{3})$.

14 From (3) the area is $\int_{-\pi/3}^{\pi/3}\frac{1}{2}(\cos^2\theta-\frac{1}{4})d\theta = [\frac{\theta}{4}+\frac{\sin 2\theta}{8}-\frac{\theta}{8}]_{-\pi/3}^{\pi/3} = \frac{1}{8}(\frac{2\pi}{3})+\frac{1}{8}(\frac{\sqrt{3}}{2}-(-\frac{\sqrt{3}}{2})) = \frac{\pi}{12}+\frac{\sqrt{3}}{8}$.

16 The spiral $r=e^{-\theta}$ starts at $r=1$ and returns to the x axis at $r=e^{-2\pi}$. Then it goes inside itself (no new area). So area $=\int_0^{2\pi}\frac{1}{2}e^{-2\theta}d\theta = [-\frac{1}{4}e^{-2\theta}]_0^{2\pi} = \frac{1}{4}(1-e^{-4\pi})$.

18 $\frac{dx}{d\theta} = -\sin\theta F(\theta)+\cos\theta\frac{dF}{d\theta}$ and similarly for $\frac{dy}{d\theta}$. Then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta\,F(\theta)+\sin\theta\,dF/d\theta}{-\sin\theta\,F(\theta)+\cos\theta\,dF/d\theta}$. Divide top and bottom by $\cos\theta$ to reach $\frac{F(\theta)+\tan\theta\,dF/d\theta}{-\tan\theta F(\theta)+dF/d\theta}$.

20 Simplify $\frac{\tan\phi-\tan\theta}{1+\tan\phi\,\tan\theta} = \frac{\frac{F+\tan\theta F'}{-\tan\theta F+F'}-\tan\theta}{1+\frac{F+\tan\theta F'}{-\tan\theta F+F'}\tan\theta} = \frac{F+\tan\theta F'-\tan\theta(-\tan\theta F+F')}{-\tan\theta F+F'+\tan\theta(F+\tan\theta F')} = \frac{(1+\tan^2\theta)F}{(1+\tan^2\theta)F'} = \frac{F}{F'}$.

22 $r=1-\cos\theta$ is the mirror image of Figure 9.4c across the y axis. By Problem 20, $\tan\psi=\frac{F}{F'} = \frac{1-\cos\theta}{\sin\theta}$. This is $\frac{\frac{1}{2}\sin^2\frac{\theta}{2}}{\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$. So $\psi=\frac{\theta}{2}$ (check at $\theta=\pi$ where $\psi=\frac{\pi}{2}$).

24 By Problem 18 $\frac{dy}{dx} = \frac{\cos\theta+\tan\theta(-\sin\theta)}{-\cos\theta\,\tan\theta-\sin\theta} = \frac{\cos^2\theta-\sin^2\theta}{\cos\theta(-2\sin\theta)} = -\frac{\cos 2\theta}{\sin 2\theta} = -\frac{1}{\sqrt{3}}$ at $\theta=\frac{\pi}{6}$. At that point $x=r\,\cos\,\theta =$ $\cos^2\frac{\pi}{6} = (\frac{\sqrt{3}}{2})^2$ and $y=r\,\sin\theta = \cos\frac{\pi}{6}\,\sin\frac{\pi}{6} = \frac{1}{2}(\frac{\sqrt{3}}{2})$. The tangent line is $y-\frac{\sqrt{3}}{4} = -\frac{1}{\sqrt{3}}(x-\frac{3}{4})$.

26 $r=\sec\theta$ has $\frac{dr}{d\theta} = \sec\theta\tan\theta$ and $\frac{ds}{d\theta} = \sqrt{\sec^2\theta+\sec^2\theta\,\tan^2\theta} = \sqrt{\sec^4\theta} = \sec^2\theta$. Then arc length $=\int_0^{\pi/4}\sec^2\theta\,d\theta = \tan\frac{\pi}{4} = 1$. Note: $r=\sec\theta$ is the line $r\,\cos\theta=1$ or $x=1$ from $y=0$ up to $y=1$.

28 $r=\theta^2$ has $\frac{dr}{d\theta}=2\theta$ and $\frac{ds}{d\theta} = \sqrt{\theta^4+4\theta^2}$. Then arc length $=\int_0^\pi\theta\sqrt{\theta^2+4}d\theta = [\frac{1}{3}(\theta^2+4)^{3/2}]_0^\pi$ $=\frac{1}{3}[(\pi^2+4)^{3/2}-4^{3/2}]$.

30 $ds=\sqrt{\cos^2\theta+\sin^2\theta}d\theta = d\theta$ and surface area $=\int 2\pi y\,ds = \int 2\pi\,r\,\sin\theta\,ds = \int_0^{\pi/2}2\pi\,\cos\theta\,\sin\theta\,d\theta = \pi$.

32 $r=1+\cos\,\theta$ has $\frac{ds}{d\theta} = \sqrt{(1+2\,\cos\,\theta+\cos^2\theta)+\sin^2\theta} = \sqrt{2+2\,\cos\theta}$. Also $y=r\,\sin\,\theta = (1+\cos\,\theta)\sin\theta$. Surface area $\int 2\pi\,y\,ds = 2\pi\sqrt{2}\int_0^\pi(1+\cos\,\theta)^{3/2}\sin\,\theta\,d\theta = [2\pi\sqrt{2}(-\frac{2}{5})(1+\cos\,\theta)^{5/2}]_0^\pi =$ $2\pi\sqrt{2}(\frac{2}{5})2^{5/2} = \frac{32\pi}{5}$.

34 $y = r\sin\theta = \sin\theta\cos\theta$ and $x = \cos^2\theta$ (which moves *left*). Volume $= \int \pi y^2 dx =$
$\int_0^{\pi/2} \pi\sin^2\theta\cos^2\theta(2\cos\theta\sin\theta)d\theta = 2\pi\int_0^{\pi/2}(\sin^3\theta - \sin^5\theta)\cos\theta\, d\theta = 2\pi[\frac{\sin^4\theta}{4} - \frac{\sin^6\theta}{6}]_0^{\pi/2} = 2\pi[\frac{1}{4} - \frac{1}{6}] = \frac{\pi}{6}$.
Check: The sphere has radius $\frac{1}{2}$ and volume $\frac{4\pi}{3}(\frac{1}{2})^3 = \frac{\pi}{6}$.

36 $r = \sec\theta$ has $x = r\cos\theta = 1$ and $ds = \sec^2\theta\, d\theta$ as in Problem 26. Surface area $=$
$\int 2\pi x ds = \int_0^{\pi/4} 2\pi(1)\sec^2\theta\, d\theta = [2\pi\tan\theta]_0^{\pi/4} = \mathbf{2\pi}$. The surface is a cylinder.

38 The triangle connecting the three centers has $60°$ angles and base 2. Its area is $\frac{1}{2}(2)(2\sin\frac{\pi}{3}) = \sqrt{3}$.
Subtract the area inside the circles and triangle: 3 times $\frac{\pi}{6}$. Remaining area $= \sqrt{3} - \frac{\pi}{2}$.

40 The parameter θ along the ellipse $x = 4\cos\theta, y = 3\sin\theta$ is *not* the angle from the origin. For example at $\theta = \frac{\pi}{4}$ the point (x, y) is *not* on the $45°$ line. So the area formula $\int \frac{1}{2}r^2 d\theta$ does not apply. The correct area is 12π.

9.4 Complex Numbers (page 364)

The complex number $3+4i$ has real part **3** and imaginary part **4**. Its absolute value is $r = \mathbf{5}$ and its complex conjugate is $\mathbf{3 - 4i}$. Its position in the complex plane is at **(3,4)**. Its polar form is $r\cos\theta + ir\sin\theta = \mathbf{re^{i\theta}}$ (or $\mathbf{5e^{i\theta}}$). Its square is $\mathbf{-7 - 14i}$. Its nth power is $\mathbf{r^n}e^{in\theta}$.

The sum of $1+i$ and $1-i$ is **2**. The product of $1+i$ and $1-i$ is **2**. In polar form this is $\sqrt{2}e^{i\pi/4}$ times $\mathbf{\sqrt{2}e^{-i\pi/4}}$. The quotient $(1+i)/(1-i)$ equals the imaginary number **i**. The number $(1+i)^8$ equals **16**. An eighth root of 1 is $w = \mathbf{(1+i)/\sqrt{2}}$. The other eighth roots are $\mathbf{w^2, w^3, \cdots, w^7, w^8 = 1}$.

To solve $d^8y/dt^8 = y$, look for a solution of the form $y = \mathbf{e^{ct}}$. Substituting and canceling e^{ct} leads to the equation $\mathbf{c^8 = 1}$. There are **eight** choices for c, one of which is $(-1+i)/\sqrt{2}$. With that choice $|e^{ct}| = \mathbf{e^{-t/\sqrt{2}}}$. The real solutions are Re $e^{ct} = \mathbf{e^{-t/\sqrt{2}}\cos\frac{t}{\sqrt{2}}}$ and Im $e^{ct} = \mathbf{e^{-t/\sqrt{2}}\sin\frac{t}{\sqrt{2}}}$.

1 Sum $= 4$, product $= 5$ **5** Angles $\frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4}$ **7** Real axis; imaginary axis; $\frac{1}{2}$ axis $x \geq 0$; unit circle
9 $cd = 5 + 10i, \frac{c}{d} = \frac{11-2i}{25}$ **11** $2\cos\theta, 1; -1, 1$ **13** Sum $= 0$, product $= -1$ **15** $r^4e^{4i\theta}, \frac{1}{r}e^{-i\theta}, \frac{1}{r^4}e^{-4i\theta}$
17 Evenly spaced on circle around origin **19** e^{it}, e^{-it} **21** e^t, e^{-t}, e^0 **23** $\cos 7t, \sin 7t$
29 $t = -\frac{2\pi}{\sqrt{3}}, y = -e^{\pi/\sqrt{3}}$ **31** F; T; at most 2; Re $c < 0$ **33** $\frac{1}{r}e^{-i\theta}, x = \frac{1}{r}\cos\theta, y = -\frac{1}{r}\sin\theta; \pm\frac{1}{\sqrt{r}}e^{-i\theta/2}$

2 $1+i$ has $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$; $(1+i)^2 = 2i$ has $r = 2$ and $\theta = \frac{\pi}{2}$; $\frac{1}{1+i} = \frac{1-i}{1-i^2} = \frac{1-i}{2}$ has $r = \frac{\sqrt{2}}{2}$ and $\theta = -\frac{\pi}{4}$.

4 The powers of $e^{2\pi i/6}$ are on the unit circle at equally spaced angles $\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$.

6 $4e^{i\pi/3} = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 4(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = \mathbf{2 + 2\sqrt{3}i}$. The square roots are $\mathbf{2e^{i\pi/6}}$ and $\mathbf{-2e^{i\pi/6}} = 2e^{7\pi i/6}$.

8 $x + iy = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ at $\theta = 45°$, $x + iy = \mathbf{i}$ at $\theta = 90°$, $x + iy = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ at $\theta = 135°$. Verify
$(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^2 = \frac{1}{2} + i + i^2(\frac{1}{2}) = i$ and then $i(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

10 $e^{ix} = i$ yields $\mathbf{x} = \frac{\pi}{2}$ (note that $\frac{i\pi}{2}$ becomes $\ln i$); $e^{ix} = e^{-1}$ yields $\mathbf{x = i}$, second solutions are $\frac{\pi}{2} + 2\pi$ and $i + 2\pi$.

12 $e^{i\theta} + e^{i\phi}$ is at the middle angle $\frac{\theta+\phi}{2}$ with length $2\cos\frac{\theta-\phi}{2}$; $e^{i\theta}$ times $e^{i\phi}$ equals $\mathbf{e^{i(\theta+\phi)}}$; $\mathbf{e^{2\pi i/3} + e^{4\pi i/3}} =$
$e^{2\pi i/3} + e^{-2\pi i/3} = 2\cos\frac{2\pi}{3} = -1$; $e^{2\pi i/3}$ times $e^{4\pi i/3}$ equals $e^{6\pi i/3} = 1$.

14 The roots of $c^2 - 4c + 5 = 0$ must multiply to give **5**. Check: The roots are $\frac{4\pm\sqrt{16-20}}{2} = 2 \pm i$. Their product is $(2+i)(2-i) = 4 - i^2 = 5$.

16 $(\cos\theta + i\sin\theta)^3 = (\cos^3\theta + 3i^2\cos\theta\sin^2\theta) + (3i\cos^2\theta\sin\theta + i^3\sin^3\theta)$. Match with $e^{3i\theta}$ to find real part $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ (or $4\cos^3\theta - 3\cos\theta$). The imaginary part is $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ (or $3\sin\theta - 4\sin^3\theta$).

18 The fourth roots of $re^{i\theta}$ are $r^{1/4}$ times $e^{i\theta/4}, e^{i(\theta+2\pi)/4}, e^{i(\theta+4\pi)/4}, e^{i(\theta+6\pi)/4}$. Multiply $(r^{1/4})^4$ to get r. Add angles to get $(4\theta + 12\pi)/4 = \theta + 3\pi$. The product of the 4 roots is $re^{i(\theta+3\pi)} = -re^{i\theta}$.

20 $(e^{ct})''' + e^{ct} = 0$ gives $(c^3+1)e^{ct} = 0$. Then $c^3 = -1 = e^{i\pi}$ and $c = \mathbf{e^{i\pi/3}}, \mathbf{e^{i\pi}}, \mathbf{e^{i5\pi/3}}$. The root $c = e^{i\pi} = -1$ gives $y = \mathbf{e^{-t}}$. The other roots give $y = e^{(1+\sqrt{3}i)t/2}$ and $y = e^{(1-\sqrt{3}i)t/2}$. (Note: Real solutions are: $y = \mathbf{e^{t/2}\cos\frac{\sqrt{3}}{2}t}$ and $y = \mathbf{e^{t/2}\sin\frac{\sqrt{3}}{2}t}$.)

22 $(e^{ct})'' + 6(e^{ct})' + 5e^{ct} = 0$ gives $c^2 + 6c + 5 = 0$ or $(c+5)(c+1) = 0$. Then $c = -5$ yields $\mathbf{y = e^{-5t}}$ and $c = -1$ yields $\mathbf{y = e^{-t}}$.

24 $c^2 - 2c + 2 = 0$ gives $c = 1 \pm i$. Then the real part of $e^{(1+i)t}$ is $y = \mathbf{e^t\cos t}$ and the imaginary part is $\mathbf{e^t\sin t}$.

26 $e^{(-1+i)t} = e^{-t}\cos t + ie^{-t}\sin t$ spirals in to $e^c = e^{-1}\cos 1 + ie^{-1}\sin 1 \approx .2 + .3i$ at $t = 1$.

28 $\frac{dy}{dt} = iy$ leads to $y = e^{it} = \cos t + i\sin t$. Matching real and imaginary parts of $\frac{d}{dt}(\cos t + i\sin t) = i(\cos t + i\sin t)$ yields $\frac{d}{dt}\cos t = -\sin t$ and $\frac{d}{dt}\sin t = \cos t$.

30 $\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)) = \frac{1}{2}(2\cos\theta) = \cos\theta$. Similarly $\sin\theta = \mathbf{\frac{1}{2i}(e^{i\theta} - e^{-i\theta})}$.

32 $re^{i\theta}$ times $Re^{i\phi}$ equals $\mathbf{(rR)e^{i(\theta+\phi)}}$. The rectangular form is $rR\cos(\theta+\phi) + irR\sin(\theta+\phi)$. This equals $(r\cos\theta + ir\sin\theta)(R\cos\phi + iR\sin\phi) = \mathbf{rR(\cos\theta\cos\phi - \sin\theta\sin\phi)} + irR(\cos\theta\sin\phi + \sin\theta\cos\phi)$.

34 Problem 30 yields $\cos ix = \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) = \frac{1}{2}(e^{-x} + e^{x}) = \cosh x$; similarly $\sin ix = \frac{1}{2i}(e^{i(ix)} - e^{-i(ix)}) = \frac{i}{2i}(e^{-x} - e^{x}) = i\sinh x$. With $x = 1$ the cosine of i equals $\frac{1}{2}(e^{-1} + e^{1}) =$ **3.086**. The cosine of i is larger than 1!