CHAPTER 2 DERIVATIVES

2.1 The Derivative of a Function (page 49)

The derivative is the limit of $\Delta f/\Delta t$ as Δt approaches **zero.** Here Δf equals $\mathbf{f(t+\Delta t)-f(t)}$. The step Δt can be positive or **negative.** The derivative is written v or **df/dt** or $\mathbf{f'(t)}$. If $f(x)=2x+3$ and $\Delta x=4$ then $\Delta f=\mathbf{8}$. If $\Delta x=-1$ then $\Delta f=\mathbf{-2}$. If $\Delta x=0$ then $\Delta f=\mathbf{0}$. The slope is not 0/0 but $df/dx=\mathbf{2}$.

The derivative does not exist where $f(t)$ has a **corner** and $v(t)$ has a **jump.** For $f(t)=1/t$ the derivative is $\mathbf{-1/t^2}$. The slope of $y=4/x$ is $dy/dx=\mathbf{-4/x^2}$. A decreasing function has a **negative** derivative. The **independent** variable is t or x and the **dependent** variable is f or y. The slope of y^2 **(is not)** $(dy/dx)^2$. The slope of $(u(x))^2$ is $\mathbf{2u(x)\,du/dx}$ by the square rule. The slope of $(2x+3)^2$ is $\mathbf{2(2x+3)2=8x+12}$.

1 (b) and (c) **3** $12+3h; 13+3h; 3; 3$ **5** $f(x)+1$ **7** -6 **9** $2x+\Delta x+1; 2x+1$
11 $\frac{4}{t+\Delta t}-\frac{4}{t}=\frac{-4}{t(t+\Delta t)}\to\frac{-4}{t^2}$ **13** 7; 9; corner **15** $A=1,\quad B=-1$ **17** F; F; T; F
19 $b=B; m$ and $M; m$ or undefined **21** Average $x_2+x_1\to 2x_1$
25 $\frac{1}{2}$; no limit (one-sided limits $1,-1$); 1; 1 if $t\neq 0$, -1 if $t=0$ **27** $f'(3); f(4)-f(3)$
29 $2x^4(4x^3)=8x^7$ **31** $\frac{du}{dx}=\frac{1}{2u}=\frac{1}{2\sqrt{x}}$ **33** $\frac{\Delta f}{\Delta x}=-\frac{1}{2}; f'(2)$ doesn't exist **35** $2f\frac{df}{dx}=4u^3\frac{du}{dx}$

2 (a) $\frac{\Delta f}{h}=\frac{2hx+h^2}{h}$ becomes $\mathbf{2x}$ at $h=0$ (b) $\frac{(x+5h)^2-x^2}{5h}=\frac{10hx+25h^2}{5h}=2x+5h$ becomes $\mathbf{2x}$ at $h=0$
(c) $\frac{(x+h)^2-(x-h)^2}{2h}=\frac{4xh}{2h}=\mathbf{2x}$ *always* (d) $\frac{(x+1)^2-x^2}{h}=\frac{2x+1}{h}\to\infty$ as $h\to 0$
4 x^2+1, x^2+10, x^2-100
6 The line and parabola have slopes 1 and $2x$. So the touching point must have $x=\frac{1}{2}$. There $y=\frac{1}{2}$ for the line, $y=(\frac{1}{2})^2+c$ for the parabola so $c=\mathbf{\frac{1}{4}}$.
8 $\frac{f(2)-f(\frac{1}{2})}{2-\frac{1}{2}}=\frac{\frac{1}{2}-2}{2-\frac{1}{2}}=-\mathbf{1}; \frac{f(1)-f(\frac{1}{2})}{1-\frac{1}{2}}=\frac{1-2}{\frac{1}{2}}=-\mathbf{2}; \frac{f(\frac{101}{200})-f(\frac{1}{2})}{\frac{101}{200}-\frac{1}{2}}=\frac{\frac{200}{101}-2}{\frac{1}{200}}=-\frac{400}{101}\approx-\mathbf{4}$.
10 $\frac{\Delta y}{\Delta x}=\frac{1+2(x+\Delta x)+3(x+\Delta x)^2-1-2x-3x^2}{\Delta x}=2+6x+3\Delta x$. Then $\frac{dy}{dx}=2+6x$.
12 $\Delta f=\frac{1}{(t+\Delta t)^2}-\frac{1}{t^2}=\frac{t^2-(t+\Delta t)^2}{t^2(t+\Delta t)^2}=\frac{-2t\Delta t-(\Delta t)^2}{t^2(t+\Delta t)^2}$. Now divide by Δt and set $\Delta t=0$: answer $\frac{-2t-0}{t^4}=-\mathbf{\frac{2}{t^3}}$.
14 $y=3x^2$ has $\frac{dy}{dx}=3$ times $2x$ and then $\frac{d^2y}{dx^2}=3$ times $2=\mathbf{6}$.
16 At $x=2$ we want $y=4$ and $\frac{dy}{dx}=B+2x=0$. So $A+2B+4=4$ and $B+2(\frac{1}{2})=0$. Then $\mathbf{B=-1}$ and $\mathbf{A=2}$.
18 $\frac{1}{x+h}-\frac{1}{x-h}=\frac{(x-h)-(x+h)}{(x+h)(x-h)}=\frac{-2h}{x^2-h^2}$. Divide by $2h$ because the centered difference went from $x-h$ to $x+h$ (an average over distance $2h$). Division by $2h$ leaves $\frac{\Delta y}{\Delta x}=\frac{-1}{x^2-h^2}$; at $h=0$ this is $\frac{dy}{dx}=\frac{-1}{x^2}$.
20 The ratios are $\frac{y(\frac{1}{4}+\frac{1}{12})-y(\frac{1}{4})}{\frac{1}{12}}=\frac{3-4}{\frac{1}{12}}=-12$ (forward difference); $\frac{y(\frac{1}{4})-y(\frac{1}{4}-\frac{1}{12})}{\frac{1}{12}}=\frac{4-6}{\frac{1}{12}}=-24$ (backward difference); $\frac{y(\frac{1}{4}+\frac{1}{12})-y(\frac{1}{4}-\frac{1}{12})}{\frac{2}{12}}=\frac{3-6}{\frac{2}{12}}=-18$ (centered difference is closest).
22 The graph of $f(t)$ has slope -2 until it reaches $t=2$ where $f(2)$ equals -1; after that it has slope zero. So f' jumps from -2 to 0 (undefined at the jump).
24 $\frac{0}{\Delta t}$ is always zero, as Δt gets smaller. The limit of zero (unchanging number) is **zero.**
26 If $\frac{f(x)}{x}$ has any limit then $f(0)$ must be zero. (In this section functions are assumed to be civilized.) Then $f'(0)$ is the limit of $\frac{f(x)-f(0)}{x-0}$, which is $\frac{f(x)}{x}$ and approaches 7. Example: $f(x)=\mathbf{7x+x^2}$.
28 By the square rule $\frac{d}{dx}(x)^2=2x(\frac{dx}{dx})=2x$
30 If $u=1$ the square rule gives $\frac{d}{dx}(1)^2=2(1)\frac{d1}{dx}$ or $\frac{d1}{dx}=2$ times $\frac{d1}{dx}$. This is possible because $\frac{d1}{dx}$ is zero and

2 times zero is zero.

32 In the figure, $f(t+\Delta t)$ is the height of the curve above $t+\Delta t$; the time step Δt is the distance from t across to $t+\Delta t$; the change Δf is the height of one red "bullet" above the other. The secant line between bullets has slope $\frac{\Delta f}{\Delta t}$. The tangent line at the lower bullet has slope $f'(t)$.

34 For $x=0$ and $\Delta x=1$ the function $f(x)=x^2-x$ has $\Delta f=f(1)-f(0)=0$. But the slope f' at $x=0$ is -1. This problem will be worded more carefully in the future.

36 (a) **False** First draw a curve that stays below $y=x$ but comes upward steeply for negative x. Then create a formula like $\mathbf{y=-x^2-10}$. (b) **False** $f(x)$ could be any constant, for example $f(x)=10$. Note what is *true*: If $\frac{df}{dx}\le 1$ and $f(x)\le x$ at some point then $f(x)\le x$ everywhere **beyond that point.**

38 For $f(x)=\frac{1}{2}x$ the graph of $f(x+h)=\frac{1}{2}(x+h)$ is above it by the vertical distance $\frac{1}{2}h$. Then $\Delta f=\frac{1}{2}h$ is a horizontal line (down near the axis!) and $\frac{\Delta f}{h}=\frac{1}{2}$ is also horizontal.

2.2 Powers and Polynomials (page 56)

The derivative of $f=x^4$ is $f'=\mathbf{4x^3}$. That comes from expanding $(x+h)^4$ into the five terms $\mathbf{x^4+4x^3h+6x^2h^2+4xh^3+h^4}$. Subtracting x^4 and dividing by h leaves the four terms, $\mathbf{4x^3+6x^2h+4xh^2+h^3}$. This is $\Delta f/h$, and its limit is $\mathbf{4x^3}$.

The derivative of $f=x^n$ is $f'=\mathbf{nx^{n-1}}$. Now $(x+h)^n$ comes from the **binomial** theorem. The terms to look for are $x^{n-1}h$, containing only one $\mathbf{h}$. There are $\mathbf{n}$ of those terms, so $(x+h)^n=x^n+\mathbf{nx^{n-1}h}+\cdots$. After subtracting $\mathbf{x^n}$ and dividing by h, the limit of $\Delta f/h$ is $\mathbf{nx^{n-1}}$. The coefficient of $x^{n-j}h^j$, not needed here, is "n choose j" $=\mathbf{n!/j!(n-j)!}$, where $n!$ means $\mathbf{n(n-1)\cdots(1)}$.

The derivative of x^{-2} is $\mathbf{-2x^{-3}}$. The derivative of $x^{1/2}$ is $\mathbf{\frac{1}{2}x^{-1/2}}$. The derivative of $3x+(1/x)$ is $\mathbf{3-1/x^2}$, which uses the following rules: the derivative of $3f(x)$ is $\mathbf{3f'(x)}$ and the derivative of $f(x)+g(x)$ is $\mathbf{f'(x)+g'(x)}$. Integral calculus recovers $\mathbf{y}$ from dy/dx. If $dy/dx=x^4$ then $y(x)=\mathbf{x^5/5}$.

1 $6x^5; 30x^4; f'''''' = 720 = 6!$ **3** $2x+7$ **5** $1+2x+3x^2+4x^3$ **7** $nx^{n-1}-nx^{-n-1}$

9 $1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$ **11** $-\frac{1}{x},(-\frac{1}{x})+5$ **13** $x^{-2/3}; x^{-4/3}; -\frac{1}{9}x^{-4/3}$

15 $3x^2-1=0$ at $x=\frac{1}{\sqrt{3}}$ and $\frac{-1}{\sqrt{3}}$ **17** 8 ft/sec; $-$ 8 ft/sec; 0 **19** Decreases for $-1<x<\frac{1}{3}$

21 $\frac{(x+h)-x}{h(\sqrt{x+h}-\sqrt{x})}\to\frac{1}{2\sqrt{x}}$ **23** 1 5 10 10 5 1 adds to $(1+1)^5 (x=h=1)$

25 $3x^2$; $2h$ is difference of x's **27** $\frac{\Delta f}{\Delta x}=2x+\Delta x+3x^2+3x\Delta x+(\Delta x)^2\to 2x+3x^2=$ sum of separate derivatives

29 $7x^6; 7(x+1)^6$ **31** $\frac{1}{24}x^4$ plus any cubic **33** $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4$ **35** $\frac{1}{24}x^4, \frac{1}{120}x^5$

37 F; F; F; T; T **39** $\frac{y}{x}=.12$ so $\frac{\Delta y}{\Delta x}=\frac{1}{2}(.12)$; six cents **41** $\frac{\Delta y}{\Delta x}=\frac{1}{\Delta x}(\frac{c}{x+\Delta x}-\frac{c}{x}), \frac{dy}{dx}=-\frac{c}{x^2}$

43 $E=\frac{2x}{2x+3}$ **45** t to $\sqrt[3]{2}t$ **47** $\frac{1}{10}x^{10}; \frac{1}{n+1}x^{n+1}$; divide by $n+1=0$

49 $.7913, -3.7913, 1.618, -.618; 0, 1.266, -2.766$

2 $f(x)=\mathbf{\frac{1}{7}x^7}$ (or $\frac{1}{7}x^7+C$) **4** $f'(x)=7(\frac{-1}{x^2})+5(\frac{-2}{x^3})=\mathbf{-\frac{7}{x^2}-\frac{10}{x^3}}$.

6 $f(x)=x^4+2x^2+1$ so $\frac{df}{dx}=4x^3+2(2x)=\mathbf{4x^3+4x}$. Or use the square rule: $\frac{df}{dx}=2(x^2+1)\frac{d}{dx}(x^2+1)=2(x^2+1)(2x)=4x^3+4x$.

8 $\frac{df}{dx}=\frac{1}{n!}(nx^{n-1})=\mathbf{\frac{x^{n-1}}{(n-1)!}}$. Note the step $\frac{n}{n(n-1)\cdots(1)}=\frac{1}{(n-1)\cdots(1)}=\frac{1}{(n-1)!}$

10 $f'(x) = \frac{2}{3}(\frac{3}{2}x^{1/2}) + \frac{2}{5}(\frac{5}{2}x^{3/2}) = x^{1/2} + x^{3/2}$.

12 First difficulty: The number of terms should be a whole number, so x is restricted to integers. Real difficulty: Increasing x not only increases each of the terms in $x + x + \cdots + x$, **it also increases the number of terms.** If x increases by 1, then $x + x + \cdots + x$ not only increases by $1 + 1 + \cdots + 1$, but also by another x (or maybe $x + 1$).

14 The slope of $x + \frac{1}{x}$ is $1 - \frac{1}{x^2}$ which is zero at $x = 1$. At that point the graph of $x + \frac{1}{x}$ levels off. (The function reaches its minimum, which is 2. For any other positive x, the combination $x + \frac{1}{x}$ is larger than 2.)

16 The function $\mathbf{f(x)} = \frac{\mathbf{1}}{\mathbf{x}}$ has a negative derivative but $f(x)$ never becomes negative. (To define $f(x)$ for all x, take $f(x) = 2 - x$ up to $x = 1$.)

18 The units of $f'$are **feet per second**; the units of f'' are $\mathbf{ft/sec^2}$. The second 16 is 16 ft/sec^2.

20 At a point where $\frac{dy}{dx} = 0$, **the tangent to the graph is horizontal.** This may be a minimum point or a maximum point; for $y = x^3$ the origin is a "pause point".

22 If $y = \frac{1}{\sqrt{x}}$ then $\Delta y = \frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}$ = (multiply top and bottom by $\sqrt{x} + \sqrt{x+h}$) = $\frac{\mathbf{x-(x+h)}}{\mathbf{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}}$. Cancel $x - x$ in the numerator and divide by h : $\frac{\Delta y}{h} = \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$.
Now let $h \to 0$ to find $\frac{dy}{dx} = \frac{-1}{2x^{3/2}} = -\frac{1}{2}x^{-3/2}$ (which is nx^{n-1}).

24 $(x + h)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$. So $\binom{5}{1} = 5$ and $\binom{5}{2} = 10$ and $\binom{5}{3} = 10$.

26 If $y'' = x$ then $y' = \frac{1}{2}x^2$ (plus any constant C). Then $y = \frac{1}{2}(\frac{1}{3}x^3)$ plus Cx plus any constant D : $\mathbf{y = \frac{1}{6}x^3 + Cx + D}$.

28 The derivative of $(u(x))^2$ is $2u(x)\frac{du}{dx}$ by the square rule. If $u = x^n$ then the derivative of x^{2n} is $\mathbf{2x^n(nx^{n-1}) = 2nx^{2n-1}}$ which follows the power rule.

30 If $\frac{df}{dx} = v(x)$ then (a) $\mathbf{4f(x)}$ has slope $4v(x)$ (b) $\mathbf{f(x) + x}$ has slope $v(x) + 1$ (c) $\mathbf{f(x+1)}$ has slope $v(x + 1)$ (d) $\mathbf{f(x) + v(x)}$ has slope $v(x) + v'(x)$

32 $\mathbf{y = \frac{x^n}{n!}}$ has $\frac{dy}{dx} = \frac{nx^{n-1}}{n!} = \frac{x^{n-1}}{(n-1)!}$ and second derivative $\frac{x^{n-2}}{(n-2)!}$ and eventually the nth derivative is 1. Check $n = 3 : y = \frac{x^3}{6}, y' = \frac{x^2}{2}, y'' = x, \mathbf{y'''} = \mathbf{1}$. Note how we are led to $\frac{x^0}{0!} = 1$.

34 If $\frac{df}{dx} = x^{-2} - x^{-3}$ then $f(x) = -1x^{-1} - \frac{1}{2}x^{-2}$ 36 $\frac{dy}{dx} = 2\sqrt{y}$ is solved by $y = x^2$ (provided $x > 0$).

38 If $y = y_0 + cx$ then $E(x) = \frac{dy/dx}{y/x} = \frac{c}{\frac{y_0}{x}+c}$ which approaches **1** as $x \to \infty$.

40 (a) High price elasticity means that the price curve steepens: as you buy more stock and get close to having a corner on the market. (b) Low price elasticity mans that the curve flattens: switch to unlimited service for making local phone calls.

42 $y = x^n$ has $E = \frac{dy/dx}{y/x} = \mathbf{n}$. The revenue $xy = x^{n+1}$ has $E = \mathbf{n + 1}$.

44 Marginal propensity to save is $\mathbf{\frac{dS}{dI}}$. Elasticity is not needed because S and I have the same **units**. Applied to the whole economy this is **macroeconomics**.

46 Relative growth of y and x is $\frac{dy/y}{dx/x}$. A child is born with relatively large head size y. Then growth of the body catches up ($\mathbf{n < 1}$).

48 In general $\frac{a^3-b^3}{a-b} = a^2 + ab + b^2$. We can directly verify $\frac{(x+h)^3-x^3}{h} = (x + h)^2 + (x + h)x + x^2$.
As $h \to 0$ this gives $\frac{dy}{dx} = 3x^2$. Similarly $\frac{a^4-b^4}{a-b} = a^3 + a^2b + ab^2 + b^3$ and directly $\frac{(x+h)^4-x^4}{h} = (x + h)^3 + (x + h)^2x + (x + h)x^2 + x^3$.

50 Two graphs touch when the difference $y_3 = \mathbf{y_1 - y_2} = x^4 + x^3 - 7x + 5$ is **zero.** At $x = 1$ we find $y_3 = 0$ (graphs touch) and also $y_3' = 4x^3 + 3x^2 - 7 = 0$ (graphs are **tangent**). The curves don't cross.

52 The expected payoff can be greater than the cost of buying a ticket for every combination. This happens when most other players have chosen from a small set of favorite "lucky" numbers. The Massachusetts lottery does have unequal popularity of different numbers, but not enough to advise buying every combination. Better to choose the unpopular numbers.

2.3 The Slope and the Tangent Line (page 63)

A straight line is determined by 2 points, or one point and the slope. The slope of the tangent line equals the slope of the curve. The point-slope form of the tangent equation is $y - f(a) = \mathbf{f'(a)(x-a)}$.

The tangent line to $y = x^3 + x$ at $x = 1$ has slope 4. Its equation is $\mathbf{y - 2 = 4(x-1)}$. It crosses the y axis at $\mathbf{y = -2}$ and the x axis at $\mathbf{x = \frac{1}{2}}$. The normal line at this point (1,2) has slope $-\frac{1}{4}$. Its equation is $y - 2 = -\frac{1}{4}\mathbf{(x-1)}$. The secant line from (1,2) to (2, 10) has slope 8. Its equation is $y - 2 = \mathbf{8(x-1)}$.

The point $(c, f(c))$ is on the line $y - f(a) = m(x-a)$ provided $m = \frac{\mathbf{f(c)-f(a)}}{\mathbf{c-a}}$. As c approaches a, the slope m approaches $\mathbf{f'(a)}$. The secant line approaches the tangent line.

1 $\frac{-12}{x^2}; y - 6 = 3(x-2); y - 6 = \frac{1}{3}(x-2); y - 6 = -\frac{3}{2}(x-2)$ **3** $y + 1 = 3(x-1); y = 3x - 4$
5 $y = x; (3,3)$ **7** $y - a = (c+a)(x-a); y - a = 2a(x-a)$ **9** $y = \frac{1}{5}x^2 + 2; y - 7 = -\frac{1}{2}(x-5)$
11 $y = 1; x = \frac{\pi}{2}$ **13** $y - \frac{1}{a} = -\frac{1}{a^2}(x-a); y = \frac{2}{a}, x = 2a$ **15** $c = 4$, tangent at $x = 2$
17 $(-3, 19)$ and $(\frac{1}{3}, \frac{13}{27})$ **19** $c = 4, y = 3 - x$ tangent at $x = 1$
21 $(1+h)^3; 3h + 3h^2 + h^3; 3 + 3h + h^2; 3$ **23** Tangents parallel, *same* normal
25 $y = 2ax - a^2, Q = (0, -a^2)$; distance $a^2 + \frac{1}{4}$; angle of incidence = angle of reflection
27 $x = 2p$; focus has $y = \frac{x^2}{4p} = p$ **29** $y - \frac{1}{\sqrt{2}} = x + \frac{1}{\sqrt{2}}; x = -\frac{2}{\sqrt{2}} = -\sqrt{2}$
31 $y - a^2 = -\frac{1}{2a}(x-a); y = a^2 + \frac{1}{2}; a = \frac{\sqrt{3}}{2}$ **33** $(\frac{1}{x^2})(1000) = 10$ at $x = 10$ hours **35** $a = 2$
37 $1.01004512; 1 + 10(.001) = 1.01$ **39** $(2+\Delta x)^3 - (8 + 6\Delta x) = 6(\Delta x)^2 + (\Delta x)^3$ **41** $x_1 = \frac{5}{4}; x_2 = \frac{41}{40}$
43 $T = 8$ sec; $f(T) = 96$ meters **45** $a = \frac{8}{5}$ meters/sec^2

2 $y = x^2 + x$ has $\frac{dy}{dx} = 2x + 1 = 3$ at $x = 1, y = 2$. The tangent line is $\mathbf{y - 2 = 3(x-1)}$ or $y = 3x - 1$. The normal line is $\mathbf{y - 2 = -\frac{1}{3}(x-1)}$ or $y = -\frac{x}{3} + \frac{7}{3}$. The secant line is $\mathbf{y - 2 = m(x-1)}$ with $m = \frac{(1+h)^2 + (1+h) - 2}{(1+h) - 1} = 3 + h$.
4 $y = x^3 + 6x$ has $\frac{dy}{dx} = 3x^2 + 6 = 6$ at $x = 0, y = 0$. The tangent line is $\mathbf{y = 6x}$. (Note how x^3 disappears.) The only crossing where $x^3 + 6x = 6x$ is at $x = 0$.
6 $x = y^2$ is $y = \sqrt{x}$ with $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$ at $x = 4$. The tangent line is $\mathbf{y - 2 = \frac{1}{4}(x-4)}$.
8 $(x-1)(x-2)$ is zero at $x = 1$ and $x = 2$. If this is the slope (it is $x^2 - 3x + 2$) then the function can be $\mathbf{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x}$. We can add any $Cx + D$ to this answer, and the slopes at $x = 1$ and 2 are still equal. $y = x^4 - 2x^2$ has $\frac{dy}{dx} = 4x^3 - 4x$. At $\mathbf{x = 1}$ **and** $\mathbf{x = -1}$ the slopes are zero and the y's are equal. The tangent line (horizontal) is the same.
10 The slope from $(a, 1/a)$ to $(c, 1/c)$ is $\frac{\frac{1}{c} - \frac{1}{a}}{c-a} = \frac{\frac{a-c}{ca}}{c-a} = -\frac{\mathbf{1}}{\mathbf{ca}}$. So the secant line has equation $\mathbf{y - \frac{1}{a} = -\frac{1}{ca}(x-a)}$. As c approaches a this becomes $\mathbf{y - \frac{1}{a} = -\frac{1}{a^2}(x-a)}$, the equation of the tangent line. Note the slope $-\frac{1}{a^2}$ for the function $y = \frac{1}{x}$.
12 If $a \to b$ and $c \to b$ then $\frac{f(c)-f(a)}{c-a}$ approaches $\mathbf{f'(b)}$, the slope at b. Test on $y = x^2$ and $y = \frac{1}{x}$.
14 If $g(x) = f(x) + 7$, the tangent lines at $x = 4$ are **parallel**. But the perpendicular(!) distance between them is **less than 7**, unless they are horizontal. (The vertical distance is 7.)
16 The problem requires $5x - 7 = x^2 + cx$ and (slopes) $5 = 2x + c$, at the same x. Then $x = \frac{5-c}{2}$. Substitute into the first equation: $5(\frac{5-c}{2}) - 7 = (\frac{5-c}{2})^2 + c(\frac{5-c}{2})$. Move all terms to the left side and simplify: $\frac{c^2}{4} - \frac{3}{4} = 0$ or $\mathbf{c = \pm\sqrt{3}}$.

18 Tangency requires $4x = cx^2$ and also (slopes) $4 = 2cx$ at the same x. The second equation gives $x = \frac{2}{c}$ and then the first is $\frac{8}{c} = \frac{4}{c}$ which has no solution.

20 The parabolas pass through $x = 1, y = 0$ if $1 + b + c = 0$ and $d - 1 = 0$. They are tangent (same slope) if $2 + b = d - 2$. Then $\mathbf{d = 1}$ and $\mathbf{b = -3}$ and $\mathbf{c = 2}$. The parabolas are $y = x^2 - 3x + 2$ and $y = 1 - x^2$.

22 The tangent line at $x = 1$ has equation $y - f(1) = f'(1)(x - 1)$. For the secant line change $f'(1)$ to $\frac{f(3)-f(1)}{3-1}$. For $f(x) = x^2 + bx + c$ (a parabola) we require $f'(1) = 2 + b$ to equal $\frac{(9+3b+c)-(1+b+c)}{3-1} = 4 + b$ (Impossible!). So try a cubic like $f(x) = x^3 + bx^2$. Then $f'(1) = 3 + 2b$ equals $\frac{(27+9b)-(1+b)}{3-1} = 13 + 4b$ if $b = -5$, which gives one possible answer $f(x) = x^3 - 5x^2$.

24 For $y = x^2 + 1$ at $x = a$ and $y = x - x^2$ at $x = c$ we require equal slopes $\mathbf{2a = 1 - 2c}$. The normal line $y - (a^2 + 1) = \frac{-1}{2a}(x - a)$ must go through the closest point $y = c - c^2$ at $x = c$. (Compare Problem 23.) Then $\mathbf{(c - c^2) - (a^2 + 1) = \frac{-1}{2a}(c - a)}$. (Final solution not required: $c - c^2 - (\frac{1}{2} - c)^2 - 1 = \frac{-1}{1-2c}(c - \frac{1}{2} + c)$ yields a cubic equation for c. Calculus will minimise (distance)^2 which involves x^4. Then derivative = 0 gives the same cubic.)

26 If a vertical ray is reflected horizontally, the tangent must go down at a 45° angle (slope -1). For $y = \frac{2}{x}$ at $x = a$ this means $\frac{dy}{dx} = \frac{-2}{a^2} = -1$ and $\mathbf{a = \sqrt{2}}$ in the figure.

28 (a) If $y = 2x$ is the tangent line at (1,2), then $\mathbf{y - 2 = -\frac{1}{2}(x - 1)}$ is the normal line. (b) As c approaches a, the secant slope $\frac{f(c)-f(a)}{c-a}$ approaches $\mathbf{f'(a)}$. (c) The line through (2,3) with slope 4 is $y - 3 = 4(x - 2)$.

30 The tangent line is $y - f(a) = f'(a)(x - a)$. This goes through $y = g(b)$ at $x = b$ if $\mathbf{g(b) - f(a) = f'(a)(b - a)}$. The slopes are the same if $\mathbf{g'(b) = f'(a)}$.

32 When the circle touches the parabola $y = \frac{x^2}{2}$ at $x = a$, the normal line has equation $y - \frac{a^2}{2} = -\frac{1}{a}(x - a)$. That line touches $x = 0$ when $y = \frac{a^2}{2} + 1$. The distance to (a, a^2) equals the radius 1 when $(a)^2 + (\frac{a^2}{2} + 1 - a^2)^2 = 1^2$. This gives $\mathbf{a = 0}$. The circle rests at the **bottom** of this flatter parabola.

34 The secant lines all have |slope| ≤ 1 so their limit the tangent line has $|\frac{df}{dx}| \le 1$. In other words $|\frac{df}{dx}(a)| = \lim_{c\to a} |\frac{f(c)-f(a)}{c-a}| \le 1$.

36 If $\frac{u(x)}{v(x)} = 7$ then $u(x) = 7v(x)$ and $u'(x) = 7v'(x)$ and $\mathbf{\frac{u'(x)}{v'(x)} = 7}$. But $(\frac{u(x)}{v(x)})' = \frac{d}{dx}(7) = \mathbf{0}$.

38 The tangent line to $y = \frac{1}{x}$ at $x = 1$ is $y - 1 = -1(x - 1)$. At $x = 1 + \Delta x$ this gives $y = 1 - \Delta x$. The curve is at height $y = \frac{1}{1+\Delta x}$. The difference is $\frac{1}{1+\Delta x} - (1 - \Delta x) = \frac{1-(1-\Delta x)(1+\Delta x)}{1+\Delta x} = \mathbf{\frac{(\Delta x)^2}{1+\Delta x}}$.

40 The distance between curve and tangent line is **of order** $\mathbf{(\Delta x)^2}$. The tangent line ignores the second derivative.

44 With acceleration changed from 3 to 2m/sec^2, Example 4 has equal speeds when $2(T-4) = V$ or $T = \frac{1}{2}V + 4$. The distance VT must equal $72 + \frac{1}{2}(2)(T - 4)^2$ when the cars meet. Then $72 + \frac{1}{4}V^2 = V(\frac{1}{2}V + 4)$ gives $0 = \frac{1}{4}V^2 + 4V - 72$ and $\mathbf{V = -8 + \sqrt{352}}$. Check: V is less than 12 because the other car is slower.

46 To *just* pass the baton, the runners reach the same point at the same time ($vt = -8 + 6t - \frac{1}{2}t^2$) and with the same speed ($v = 6 - t$). Then $(6 - t)t = -8 + 6t - \frac{1}{2}t^2$ and $\frac{1}{2}t^2 - 8 = 0$. Then $\mathbf{t = 4}$ and $\mathbf{v = 2}$.

2.4 The Derivative of the Sine and Cosine (page 70)

The derivative of $y = \sin x$ is $y' = \mathbf{\cos x}$. The second derivative (the derivative of the derivative) is $y'' = \mathbf{-\sin x}$. The fourth derivative is $y'''' = \mathbf{\sin x}$. Thus $y = \sin x$ satisfies the differential equations $y'' = \mathbf{-y}$ and $y'''' = \mathbf{y}$. So does $y = \cos x$, whose second derivative is $\mathbf{-\cos x}$.

All these derivatives come from one basic limit: $(\sin h)/h$ approaches **1**. The sine of .01 radians is very close to **.01.** So is the **tangent** of .01. The cosine of .01 is not .99, because $1-\cos h$ is much **smaller** than h. The ratio $(1-\cos h)/h^2$ approaches $\frac{1}{2}$. Therefore $\cos h$ is close to $1-\frac{1}{2}h^2$ and $\cos .01 \approx$ **.99995**. We can replace h by x.

The differential equation $y'' = -y$ leads to **oscillation.** When y is positive, y'' is **negative.** Therefore y' is **decreasing.** Eventually y goes below zero and y'' becomes **positive.** Then y' is **increasing.** Examples of oscillation in real life are **springs** and **heartbeats.**

1 (a) and (b) **3** $0; 1; 5; \frac{1}{5}$ **5** $\sin(x+2\pi); (\sin h)/h \to 1; 2\pi$ **7** $\cos^2\theta \approx 1-\theta^2+\frac{1}{4}\theta^4; \frac{1}{4}\theta^4$ is small
9 $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$ **11** $\frac{3}{2}; 4$ **13** $PS = \sin h$; area $OPR = \frac{1}{2}\sin h <$ curved area $\frac{1}{2}h$
15 $\cos x = 1 - \frac{x^2}{2\cdot 1} + \frac{x^4}{4\cdot 3\cdot 2\cdot 1} - \cdots$ **17** $\frac{1}{2h}(\cos(x+h) - \cos(x-h)) = \frac{1}{h}(-\sin x \sin h) \to -\sin x$
19 $y' = \cos x - \sin x = 0$ at $x = \frac{\pi}{4} + n\pi$ **21** $(\tan h)/h = \sin h/h\cos h < \frac{1}{\cos h} \to 1$
23 Slope $\frac{1}{2}\cos\frac{1}{2}x = \frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}$; no **25** $y = 2\cos x + \sin x; y'' = -y$ **27** $y = -\frac{1}{3}\cos 3x; y = \frac{1}{3}\sin 3x$
29 In degrees $(\sin h)/h \to 2\pi/360 = .01745$ **31** $2\sin x\cos x + 2\cos x(-\sin x) = 0$

2 (a) $\frac{h}{\sin h} \to \mathbf{1}$ (b) $(\frac{\sin h}{h})^2 \to 1^2 = \mathbf{1}$. (c) $\frac{\sin h}{\sin 2h} = \frac{1}{2}\frac{\sin h}{h}\frac{2h}{\sin 2h} \to \frac{\mathbf{1}}{\mathbf{2}}$ (d) $\frac{\sin(-h)}{h} = -\frac{\sin h}{h} \to \mathbf{-1}$.
4 $\tan h = 1.01h$ at $h = 0$ and $h = \pm.17$; $\tan h = h$ at $h = 0$.
8 .995004 versus .995; .8776 versus .875; .866 versus .863; .9986295 versus .9986292.
10 (a) $\frac{1-\cos h}{h^2} = \frac{1-\cos^2 h}{(1+\cos h)h^2} = \frac{1}{1+\cos h}(\frac{\sin h}{h})^2 \to \frac{\mathbf{1}}{\mathbf{2}}$ (b) $\frac{1-\cos^2 h}{h^2} = (\frac{\sin h}{h})^2 \to \mathbf{1}$ (c) $\frac{1-\cos^2 h}{\sin^2 h} = \mathbf{1}$
(d) $\frac{1-\cos 2h}{h} = 2\frac{1-\cos 2h}{2h} \to 2(0) = \mathbf{0}$.
12 (a) $\frac{dy}{dx}(0) = \frac{\tan h - \tan 0}{h} = \frac{\sin h}{h(\cos h)} \to \frac{1}{1} = \mathbf{1}$ (b) $\frac{dy}{dx}(0) = \frac{\sin(-h)-\sin(-0)}{h} = -\frac{\sin h}{h} \to \mathbf{-1}$
14 The slopes of $\cos x$ and $1-\frac{1}{2}x^2$ are $-\sin x$ and $\mathbf{-x}$ (close for small x). The slopes of $\sin x$ and $\mathbf{x} - \frac{\mathbf{1}}{\mathbf{6}}\mathbf{x}^{\mathbf{3}}$ (close for small x) are $\cos x$ and $1-\frac{1}{2}x^2$.
16 $\frac{\sin(x+h)-\sin(x-h)}{2h} = \frac{(\sin x\cos h + \cos x \sin h) - (\sin x \cos h - \cos x \sin h)}{2h} = \frac{\mathbf{2\cos x \sin h}}{\mathbf{2h}} \to \mathbf{\cos x}$.
18 (a) $y - \sin 0 = (\cos 0)(x-0)$ or $\mathbf{y = x}$ (tangent is 45° line) (b) $y - \sin\pi = (\cos\pi)(x-\pi)$ or $\mathbf{y = -x + \pi}$
(c) $y - \sin\frac{\pi}{4} = (\cos\frac{\pi}{4})(x - \frac{\pi}{4})$ or $\mathbf{y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(x - \frac{\pi}{4})}$.
20 (a) $\sin(x+1) = \sin x\cos 1 + \cos x \sin 1$. The derivative is $\cos x\cos 1 - \sin x \sin 1$ which is $\mathbf{\cos(x+1)}$.
(b) $\frac{\Delta y}{\Delta x} = \frac{\sin(x+1+\Delta x) - \sin(x+1)}{\Delta x} = \frac{\sin(X+\Delta x) - \sin X}{\Delta x} \to \cos X = \mathbf{\cos(x+1)}$.
22 $\frac{\sin 2(x+h) - \sin 2x}{h} = \frac{\sin 2x(\cos 2h - 1) + \cos 2x \sin 2h}{h}$. Then $\frac{\cos 2h - 1}{h} \to 0$ and $\frac{\sin 2h}{h} = 2\frac{\sin 2h}{2h} \to 2$. So the limit is $\frac{dy}{dx} = 0 + \mathbf{2\cos 2x}$.
24 The maximum of $y = \sin x + \sqrt{3}\cos x$ is at $\mathbf{x} = \frac{\boldsymbol{\pi}}{\mathbf{6}}$ (or 30°) where $\mathbf{y} = \frac{\mathbf{1}}{\mathbf{2}} + \sqrt{\mathbf{3}}\frac{\sqrt{\mathbf{3}}}{\mathbf{2}} = \mathbf{2}$. The slope at that point is $\cos x - \sqrt{3}\sin x = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \mathbf{0}$. Note that y is the same as $2\cos x$ shifted to the right by $\frac{\pi}{6}$.
26 (a) **False** (use the square rule) (b) **True** (because $\cos(-x) = \cos x$) (c) **False** for $y = x^2$ (happens to be true for $y = \sin x$) (d) **True** (y'' = slope of y' = positive when y' increases)
28 $y = \sin 5x$ has $y'' = -25\sin 5x$ so y satisfies the equation $\mathbf{y'' = -25y}$. (In general $y = \sin kx$ satisfies $y'' = -k^2y$.)
30 $\frac{dy}{dx}(\pi)$ = limit of $\frac{y(\pi+\Delta x) - y(\pi)}{\Delta x}$. For $y = \sin x$ and $\Delta x = .01$ the ratio is $\frac{\sin(\pi+.01)}{.01} = \frac{-\sin .01}{.01} = \mathbf{-.99998}$.

32 Oscillation: Volume of air in the lungs (not simple harmonic).

2.5 The Product and Quotient and Power Rules (page 77)

The derivatives of $\sin x \cos x$ and $1/\cos x$ and $\sin x/\cos x$ and $\tan^3 x$ come from the **product rule, reciprocal rule, quotient rule,** and **power rule**. The product of $\sin x$ times $\cos x$ has $(uv)' = uv' + \mathbf{u'v} = \mathbf{\cos^2 x - \sin^2 x}$. The derivative of $1/v$ is $\mathbf{-v'/v^2}$, so the slope of sec x is $\mathbf{\sin x/\cos^2 x}$. The derivative of u/v is $\mathbf{(vu' - uv')/v^2}$ so the slope of $\tan x$ is $\mathbf{(\cos^2 x + \sin^2 x)/\cos^2 x = \sec^2 x}$. The derivative of $\tan^3 x$ is $\mathbf{3\tan^2 x \sec^2 x}$. The slope of x^n is $\mathbf{nx^{n-1}}$ and the slope of $(u(x))^n$ is $\mathbf{nu^{n-1}du/dx}$. With $n = -1$ the derivative of $(\cos x)^{-1}$ is $\mathbf{-1(\cos x)^{-2}(-\sin x)}$, which agrees with the rule for $\sec x$.

Even simpler is the rule of **linearity**, which applies to $au(x) + bv(x)$. The derivative is $\mathbf{au'(x) + bv'(x)}$. The slope of $3\sin x + 4\cos x$ is $\mathbf{3\cos x - 4\sin x}$. The derivative of $(3\sin x + 4\cos x)^2$ is $\mathbf{2(3\sin x + 4\cos x)(3\cos x - 4\sin x)}$. The derivative of $\mathbf{\sin^4 x}$ is $4\sin^3 x \cos x$.

1 $2x$ **3** $\frac{-1}{(1+x)^2} - \frac{\cos x}{(1+\sin x)^2}$ **5** $(x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$
7 $-x^2 \sin x + 4x\cos x + 2\sin x$ **9** $2x - 1 - \frac{1}{\sin^2 x}$ **11** $2\sqrt{x}\sin x\cos x + \frac{1}{2}x^{-1/2}\sin^2 x + \frac{1}{2}(\sin x)^{-1/2}\cos x$
13 $4x^3\cos x - x^4 \sin x + \cos^4 x - 4x\cos^3 x \sin x$ **15** $\frac{1}{2}x^2\cos x$ **17** 0 **19** $-\frac{8}{3}(x-5)^{-5/3} + \frac{8}{3}(5-x)^{-5/3}(= 0?)$
21 $3(\sin x\cos x)^2(\cos^2 x - \sin^2 x) + 2\cos 2x$ **23** $u'vwz + v'uwz + w'uvz + z'uvw$ **25** $-\csc^2 x - \sec^2 x$
27 $V = \frac{t\cos t}{1+t}, V' = \frac{\cos t - t\sin t - t^2 \sin t}{(1+t)^2}$ $A = 2(\frac{t}{t+1} + t\cos t + \frac{\cos t}{t+1})$ $A' = 2(\cos t - t\sin t + \frac{1-\cos t}{(t+1)^2} - \frac{\sin t}{t+1})$
29 $10t$ for $t < 10$, $\frac{50}{\sqrt{t-10}}$ for $t > 10$ **31** $\frac{2t^3+3t^2}{(1+t)^2}; \frac{2t^3+6t^2+6t}{(1+t)^3}$
33 $u''v + 2u'v' + uv''; u'''v + 3u''v' + 3u'v'' + v'''$ **35** $\frac{1}{2}\sin^2 t; \frac{1}{2}\tan^2 t; \frac{2}{3}[(1+t)^{3/2} - 1]$
39 T; F; F; T; F **41** degree $2n-1$/ degree $2n$ **43** $v(t) = \cos t - t\sin t (t \le \frac{\pi}{2}); v(t) = -\frac{\pi}{2}(t \ge \frac{\pi}{2})$
45 $y = \frac{2hx^3}{L^3} + \frac{3hx^2}{L^2}$ has $\frac{dy}{dx} = 0$ at $x = 0$ (no crash) and at $x = -L$ (no dive). Then $\frac{dy}{dx} = \frac{6Vh}{L}(\frac{x^2}{L^2} + \frac{x}{L})$ and $\frac{d^2y}{dx^2} = \frac{6V^2h}{L^2}(\frac{2x}{L} + 1)$.

2 $\frac{dy}{dx} = (x^2+1)(2x) + (x^2-1)(2x) = \mathbf{4x^3}$ **4** $\mathbf{\frac{-2x}{(1+x^2)^2} + \frac{-(-\cos x)}{(1-\sin x)^2}}$.
6 $(x-1)^2 2(x-2) + (x-2)^2 2(x-1) = 2(x-1)(x-2)(x-1+x-2) = \mathbf{2(x-1)(x-2)(2x-3)}$.
8 $x^{1/2}(1+\cos x) + (x + \sin x)\frac{1}{2}x^{-1/2}$ or $\frac{3}{2}x^{1/2} + x^{1/2}\cos x + \frac{1}{2}x^{-1/2}\sin x$
10 $\frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2} + \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \mathbf{\frac{-4x}{(x^2-1)^2} + \frac{1}{\cos^2 x}}$.
12 $x^{3/2}(3\sin^2 x\cos x) + \frac{3}{2}x^{1/2}\sin^3 x + \frac{3}{2}(\sin x)^{1/2}\cos x$
14 $\sqrt{x}(\sqrt{x}+1)\frac{1}{2}x^{-1/2} + \sqrt{x}(\sqrt{x}+2)\frac{1}{2}x^{-1/2} + (\sqrt{x}+1)(\sqrt{x}+2)\frac{1}{2}x^{-1/2} = (3x + 6\sqrt{x} + 2)\frac{1}{2}x^{-1/2}$ (or other form).
16 $10(x-6)^9 + 10\sin^9 x\cos x$.
18 $\csc^2 x - \cot^2 x = \frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} = 1$ so the derivative is **zero**.
20 $\frac{(\sin x+\cos x)(\cos x+\sin x) - (\sin x - \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{2\sin^2 x + 2\cos^2 x}{(\sin x+\cos x)^2} = \mathbf{\frac{2}{(\sin x+\cos x)^2}}$
22 $\frac{x\cos x}{\sin x}$ has derivative $\frac{\sin x(-x\sin x + \cos x) - x\cos x(\cos x)}{\sin^2 x} = \frac{-x + \sin x\cos x}{\sin^2 x}$ (or other form).
24 $[u(x)]^2(2v(x)\frac{dv}{dx}) + [v(x)]^2(2u(x)\frac{du}{dx})$
26 $x\cos x + \sin x - \sin x = \mathbf{x\cos x}$ (we now have a function with derivative $x\cos x$).
28 The three slabs have volume $uv\Delta w$ and $uw\Delta v$ and $vw\Delta u$.
30 (a) Volume $= \pi r^2 h = \frac{\pi t^4}{(1+t^{3/2})^2(1+t)}$ has rate of change $\frac{(1+t^{3/2})^2(1+t)4\pi t^3 - \pi t^4(1+t^{3/2})^2 - \pi t^4(1+t)2(1+t^{3/2})\frac{3}{2}t^{1/2}}{(1+t^{3/2})^4(1+t)^2}$
(b) Surface area $= 2\pi rh + 2\pi r^2 = \frac{2\pi t^{5/2}}{(1+t^{3/2})(1+t)} + \frac{2\pi t^3}{(1+t^{3/2})^2} = \frac{2\pi t^{5/2} + 4\pi t^4 + 2\pi t^3}{(1+t^{3/2})^2(1+t)}$ has derivative

$\frac{(1+t^{3/2})^2(1+t)(5\pi t^{3/2}+16\pi t^3+6\pi t^2)-(2\pi t^{5/2}+4\pi t^4+2\pi t^3)[(1+t^{3/2})^2+(1+t)2(1+t^{3/2})\frac{3}{2}t^{1/2}]}{(1+t^{3/2})^4(1+t)^2}$.

This is a workout that you might or might not assign.

32 The derivative of $u(x)u^2(x)$ is $u(x)(2u(x)\frac{du}{dx}) + u^2(x)\frac{du}{dx} = \mathbf{3u^2(x)\frac{du}{dx}}$. This is the power rule for $u^3(x)$.

34 (a) $y = \frac{1}{4}x^4$ (b) $y = -\frac{1}{2}x^{-2}$ (c) $y = -\frac{2}{5}(1-x)^{5/2}$ (This one is more difficult.) (d) $y = -\frac{1}{3}\cos^3 x$

36 $\frac{u^3}{u^2}$ has derivative $\frac{u^2(3u^2\frac{du}{dx})-u^3(2u\frac{du}{dx})}{u^4} = \frac{u^4\frac{du}{dx}}{u^4} = \mathbf{\frac{du}{dx}}$. Then $-\frac{v'}{v^2}$ has derivative $\frac{-v^2v''+v'(2vv')}{v^4} = -\frac{v''}{v^2} + \frac{2(v')^2}{v^3}$.

38 $\mathbf{u = x - 1}$ and $\mathbf{v = x}$ have $\frac{du}{dx} = \frac{dv}{dx} = 1$ but $\frac{d}{dx}(\frac{x-1}{x}) = \frac{x-(x-1)}{x^2} = \frac{1}{x^2}$. This is positive so $\frac{u}{v}$ is increasing.

40 $\frac{d}{dt}(uv)$ has dimension $\frac{\text{shares}(\frac{\text{dollars}}{\text{share}})}{\text{time}}$. So does $u\frac{dv}{dt} = \text{shares}\,\frac{\frac{\text{dollars}}{\text{share}}}{\text{time}}$ and $v\frac{du}{dt} = \frac{\text{dollars}}{\text{share}}\frac{\text{shares}}{\text{time}}$.

42 Generally $(\frac{dy}{dx})^2$ is completely different from $\frac{d^2y}{dx^2}$. For $y = 5x+3$ they are $\mathbf{(5)^2}$ and **zero**.

2.6 Limits (page 84)

The limit of $a_n = (\sin n)/n$ is **zero**. The limit of $a_n = n^4/2^n$ is **zero**. The limit of $a_n = (-1)^n$ is **not defined**. The meaning of $a_n \to 0$ is: Only **finitely many** of the numbers $|a_n|$ can be **greater than** ϵ **(an arbitrary positive number)**. The meaning of $a_n \to L$ is: For every ϵ there is an **N** such that $\mathbf{|a_n - L|} < \epsilon$ if $n > \mathbf{N}$. The sequence $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \cdots$ is not **convergent** because eventually those sums go past **any number L**.

The limit of $f(x) = \sin x$ as $x \to a$ is $\mathbf{\sin a}$. The limit of $f(x) = x/|x|$ as $x \to -2$ is $\mathbf{-1}$, but the limit as $x \to 0$ does not **exist**. This function only has **one-sided** limits. The meaning of $\lim_{x\to a} f(x) = L$ is: For every ϵ there is a δ such that $|f(x) - L| < \epsilon$ whenever $\mathbf{0 < |x - a|} < \delta$.

Two rules for limits, when $a_n \to L$ and $b_n \to M$, are $a_n + b_n \to \mathbf{L + M}$ and $a_nb_n \to \mathbf{LM}$. The corresponding rules for functions, when $f(x) \to L$ and $g(x) \to M$ as $x \to a$, are $\mathbf{f(x) + g(x) \to L + M}$ and $\mathbf{f(x)g(x) \to LM}$. In all limits, $|a_n - L|$ or $|f(x) - L|$ must eventually go below and **stay below** any positive **number** ϵ.

$A \Rightarrow B$ means that A is a **sufficient** condition for B. Then B is true if A is true. $A \Leftrightarrow B$ means that A is a **necessary and sufficient** condition for B. Then B is true **if and only if** A is true.

1 $\frac{1}{4}, L = 0$, after $N = 10$; $\frac{25}{24}, \infty$, no N; $\frac{1}{4}, 0$, after 5; 1.1111, $\frac{10}{9}$, all n; $\sqrt{2}, 1$, after 38; $\sqrt{20} - 4, \frac{1}{2}$, all n; $\frac{625}{256}, e = 2.718\cdots$, after $N = 12$. **3** (c) and (d)

5 Outside any interval around zero there are only a finite number of a's **7** $\frac{5}{2}$ **9** $\frac{f(h)-f(0)}{h}$ **11** 1

13 1 **15** sin 1 **17** No limit **19** $\frac{1}{2}$ **21** Zero if $f(x)$ is continuous at a **23** 2

25 .001, .0001, .005, .1 **27** $|f(x) - L|$; $\frac{4x}{1+x}$ **29** $0; X = 100$ **33** $4; \infty; 7; 7$ **35** 3; no limit; 0; 1

37 $\frac{1}{1-r}$ if $|r| < 1$; no limit if $|r| \geq 1$ **39** .0001; after $N = 7$ (or 8?) **41** $\frac{1}{2}$

43 $9; 8\frac{1}{2}; a_n - 8 = \frac{1}{2}(a_{n-1} - 8) \to 0$

45 $a_n - L \leq b_n - L \leq c_n - L$ so $|b_n - L| < \epsilon$ if $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$

2 (a) is false when $L = 0 : \mathbf{a_n = \frac{1}{n}} \to 0$ and $\mathbf{b_n = \frac{1}{n^2}} \to 0$ but $\frac{a_n}{b_n} = n \to \infty$ (b) It is true that: If $a_n \to L$ then $a_n^2 \to L^2$. It is false that: If $a_n^2 \to L^2$ then $a_n \to L$: $\mathbf{a_n}$ **could approach** $-L$ or $a_n = L, -L, L, -L, \cdots$ has no limit. (c) $a_n = -\frac{1}{n}$ is negative but the limit $L = 0$ is not negative (d) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \cdots$ has infinitely many a_n in every strip around zero but a_n does not approach zero.

4 (a) $[a_n \to 1] \leftrightarrow [-a_n \to -1]$ (b) $[a_n \to 0] \Rightarrow [a_n - a_{n-1} \to 0]$ (c) $[a_n \le n] \Leftarrow [a_n = n]$
(d) $[a_n \to 0] \Rightarrow [\sin a_n \to 0]$ (e) $[a_n \to 0] \Rightarrow [\frac{1}{a_n}$ fails to converge] (f) **neither implication.**
6 Given any $\epsilon > 0$, there are δ_1 and δ_2 such that $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta_1$ and $|g(x) - M| < \epsilon$ if $0 < |x - a| < \delta_2$. **Take** $\delta =$ **smaller of** δ_1 **and** δ_2 **and add** $:|f(x) + g(x) - L - M| < \epsilon + \epsilon$ if $0 < |x - a| < \delta$.
8 No limit 10 Limits equals **f′(1) if the derivative exists.** 12 $\frac{2x \tan x}{\sin x} = \frac{2x}{\cos x} \to \frac{0}{1} = 0$
14 $|x| = -x$ when x is negative; the limit of $\frac{-x}{x}$ is 1. 16 $\frac{f(c)-f(a)}{c-a} \to$ **f′(a) if the derivative exists.**
18 $\frac{x^2-25}{x-5} = x + 5$ approaches 10 as $x \to 5$ 20 $\frac{\sqrt{4-x}}{\sqrt{6+x}}$ approaches $\frac{\sqrt{2}}{\sqrt{8}} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$ as $x \to 2$
22 $\sec x - \tan x = \frac{1-\sin x}{\cos x} = \frac{1-\sin x}{\cos x}\left(\frac{1+\sin x}{1+\sin x}\right) = \frac{1-\sin^2 x}{\cos x(1+\sin x)} = \frac{\cos x}{1+\sin x}$ which approaches $\frac{0}{2} = 0$ at $x = \frac{\pi}{2}$.
24 $\frac{\sin(x-1)}{x-1}\left(\frac{1}{x+1}\right)$ approaches $1 \cdot \frac{1}{2} = \frac{1}{2}$ as $x \to 1$ 26 Statement (2) is the definition of a limit.
28 Given any $\epsilon > 0$ there is an X such that $|f(x)| < \epsilon$ if $x < X$.
30 $|f(x) - 2| < \epsilon$ means $|\frac{2x}{1+x} - 2| < .01$ or $|\frac{2x-2-2x}{1+x}| < .01$ or $2 < .01|1 + x|$. This is true for $x > 199$.
32 The limit is $e = 2.718\cdots$
34 (a and b) $\frac{6x^3+1000x}{x \text{ or } x^2} \to \infty$ (no limit) (c and d) $\frac{6x^3+1000x}{x^3} \to 6$ as $x \to \infty$ or $x \to -\infty$.
36 The range of x is $0 < |x - a| < \delta$. If δ is reduced the range becomes smaller. So it remains true that $|f(x) - L| < \epsilon$ for all allowed x.
38 There is an N such that $|a_n - L| < \epsilon$ for $n > N$. Also $|a_m - L| < \epsilon$ for $m > N$ (and thus $|L - a_m| < \epsilon$).
Now add: $|(\mathbf{a_n} - \mathbf{L}) + (\mathbf{L} - \mathbf{a_m})| < \epsilon + \epsilon$ or $|a_n - a_m| < 2\epsilon$.
40 (a) $.493999\cdots$ approaches $L = .494$. (b) With a simple pattern the professor will find L. With random choice there is no hope. Maybe try $.49301101\cdots$ with 1's in the 2nd, 3rd, 5th, and all prime number positions. The limit requires $\sum(.1)^{\text{prime}} =$ unknown?
42 The average L has "$\frac{1}{2}$" in each decimal position: $L = .\frac{1}{2}\frac{1}{2}\frac{1}{2}\cdots = \frac{1}{2}(.111\cdots) = \frac{1}{18}$. Second method: The first digit could be 0 or 1 (average $\frac{1}{20}$). After that is another random sequence with average $\frac{1}{10}L$, since it is shifted by one decimal. So the average $\frac{1}{20} + \frac{1}{10}L$ is the same as L and $\frac{1}{20} + \frac{1}{10}L = L$ yields $L = \frac{1}{18}$.
44 For every δ the number $\epsilon = 2$ has the required (and silly) property: $|\cos x| < 2$ if $|x| < \delta$.

2.7 Continuous Functions (page 89)

Continuity requires the limit of $f(x)$ to exist as $x \to a$ and to agree with **f(a)**. The reason that $x/|x|$ is not continuous at $x = 0$ is : **it jumps from −1 to 1.** This function does have **one-sided limits.** The reason that $1/\cos x$ is discontinuous at $\mathbf{x} = \pi/\mathbf{2}$ is **that it approaches infinity.** The reason that $\cos(1/x)$ is discontinuous at $x = 0$ is **infinite oscillation.** The function $f(x) = \frac{1}{x-3}$ has a simple pole at $x = 3$, where f^2 has a **double** pole.

The power x^n is continuous at all x provided n is **positive.** It has no derivative at $x = 0$ when n **is between 0 and 1.** $f(x) = \sin(-x)/x$ approaches **−1** as $x \to 0$, so this is a **continuous** function provided we define $f(0) = \mathbf{-1}$. A "continuous function" must be continuous at all **points in its domain.** A "continuable function" can be extended to every point x so that it **is continuous.**

If f has a derivative at $x = a$ then f is necessarily **continuous** at $x = a$. The derivative controls the speed at which $f(x)$ approaches **f(a)**. On a closed interval $[a, b]$, a continuous f has the **extreme** value property and the **intermediate** value property. It reaches its **maximum** M and its **minimum** m, and it takes on every value

in between.

1 $c = \sin 1$; no c **3** Any $c; c = 0$ **5** $c = 0$ or 1; no c **7** $c = 1$; no c **9** no c; no c
11 $c = \frac{1}{64}; c = \frac{1}{64}$ **13** $c = -1; c = -1$ **15** $c = 1; c = 1$ **17** $c = -1; c = -1$
19 $c = 2, 1, 0, -1, \cdots$; same c **21** $f(x) = 0$ except at $x = 1$ **23** $\sqrt{x-1}$ **25** $-\frac{2x}{|x|}$ **27** $\frac{5}{x-1}$
29 One; two; two **31** No; yes; no **33** $xf(x), (f(x))^2, x, f(x), 2(f(x) - x), f(x) + 2x$ **35** F; F; F; T
37 Step; $f(x) = \sin \frac{1}{x}$ with $f(0) = 0$ **39** Yes; no; no; yes ($f_4(0) = 1$)
41 $g(\frac{1}{2}) = f(1) - f(\frac{1}{2}) = f(0) - f(\frac{1}{2}) = -g(0)$; zero is an intermediate value between $g(0)$ and $g(\frac{1}{2})$
43 $f(x) - x$ is ≥ 0 at $x = 0$ and ≤ 0 at $x = 1$

2 $c = \cos^3 \pi = -1$. Then the function is (A) continuous and (B) differentiable.
4 $c = 0$ gives $f(x) = 0$: both properties (A) and (B)
6 $c = -2$ gives $f(x) = x^3$: both properties (A) and (B)
8 $c > 0$ gives $f(x) = x^c$: For $0 < c < 1$ this is not differentiable at $x = 0$ but is continuous for ($x \geq 0$). For $c \geq 1$ this is continuous and differentiable where it is defined ($x \geq 0$ for noninteger c).
10 Need $x + c = 1$ at $x = c$ which gives $2c = 1$ or $\mathbf{c} = \frac{1}{2}$. Then $x + \frac{1}{2}$ matches 1 at $x = \frac{1}{2}$ (continuous but not differentiable).
12 $\mathbf{c} = 1$ gives continuity at $x = 0$. However $\sec x$ is not defined for all $x \geq 0$, which spoils (A) and (B).
14 $\mathbf{c} = 1$ gives $f(x) = \frac{x^2-1}{x-1} = x + 1$ which agrees with $2c = 2$ at $x = 1$ (continuous but **not** differentiable).
16 At $x = c$ continuity requires $c^2 = 2c$. Then $\mathbf{c} = \mathbf{0}$ **or** $\mathbf{2}$. At $x = c$ the derivative jumps from $2x$ to 2.
18 $|x + c|$ is continuous, but not differentiable, at $x = -c$ (slope jumps from -1 to 1).
20 $|x^2 + c^2|$ is continuous and differentiable at $x = -c$ (slope jumps from -1 to 1).
22 $\cos \frac{1}{x}$ **24** $\frac{1}{(x-5)^2}$ **26** $f(x) = |x - 1|^{-1/2}$
28 (a) Choose $\epsilon = 1$ (or any ϵ less than 4). There is no δ such that $|3x - 7| < 1$ when x is within δ of 1. (b) $|3x - 3| < \frac{1}{2}$ if $|x - 1| < \frac{1}{6}$. So take $\delta = \frac{1}{6}$.
30 (a) One-sided limits: $\frac{|x|}{7x} \to -\frac{1}{7}$ as $x = 0^-$ and $\frac{|x|}{7x} \to \frac{1}{7}$ as $x \to 0^+$. (b) $\sin |x|$ has a two-sided limit at $x = 0$. (c) $|x^2 - 1|$ has a sharp corner at $x = 1$ and -1. It equals $1 - x^2$ between $x = -1$ and $x = 1$. The slope changes from $2x$ to $-2x$ and back to $2x$. One-sided limits at $x = 1$ and -1.
32 Use $|\sin \frac{1}{x}| < 1$. Then (a) $x^2 \sin \frac{1}{x} \to 0$ as $x \to 0$ (b) $\frac{x^2 \sin \frac{1}{x} - 0}{x - 0} \to 0$ as $x \to 0$. (c) $f'(x) = x^2(\cos \frac{1}{x})(-\frac{1}{x^2}) + (\sin \frac{1}{x})(2x) = -\cos \frac{1}{x} + 2x \sin \frac{1}{x}$ has no limit as $x \to 0$. (Part (c) needs the chain rule or careful limits. Main point: $f'(x)$ has no limit as $x \to 0$ even though $f'(0) = 0$)
34 $f(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$ is discontinuous but $f^2(x) = 1$.
36 $\cos x$ is greater than $2x$ at $x = 0$; $\cos x$ is less than $2x$ at $x = 1$. The continuous function $\cos x - 2x$ changes from positive to negative. By the **intermediate value theorem** there is a point where $\cos x - 2x = 0$.
38 $x \sin \frac{1}{x}$ approaches zero as $x \to 0$ (so it is continuous) because $|\sin \frac{1}{x}| < 1$. There is no derivative because $\frac{f(h)-f(0)}{h} = \frac{h}{h} \sin \frac{1}{h} = \sin \frac{1}{h}$ has no limit (infinite oscillation).
40 A continuous function is continuous at each point x in its domain (where $f(x)$ is defined). A **continuable** function can be defined at all other points x in such a way that it is continuous there too. $f(x) = \frac{1}{x}$ is continuous away from $x = 0$ but not continuable.
42 $f(x) = x$ if x is a fraction, $f(x) = 0$ otherwise
44 Suppose L is the limit of $f(x)$ as $x \to a$. To prove continuity we have to show that $f(a) = L$. For any ϵ we can obtain $|f(x) - L| < \epsilon$, and this applies at $\mathbf{x} = \mathbf{a}$ (since that point is not excluded any more). Since ϵ is arbitrarily small we reach $f(a) = L$: the function has the right value at $x = a$.