CHAPTER 16 MATHEMATICS AFTER CALCULUS

16.1 Linear Algebra (page 602)

If $Ax = b$ and $AX = B$, then A times $2x + 3X$ equals **2b+3B**. If $Ax = 0$ and $AX = 0$ then A times $2x + 3X$ equals **0**. In this case x and X are in the **nullspace** of A, and so is the combination **2x + 3X**. The nullspace contains all solutions to **Ax = 0**. It is a subspace, which means **that when x and X are in the nullspace, so are all combinations ax + bX**. If $x = (1, 1, 1)$ is in the nullspace then the columns add to **the zero column**, so they are dependent.

Another subspace is the **column** space of A, containing all combinations of the columns of A. The system $Ax = b$ can be solved when b **is in the column space**. Otherwise the best solution comes from $A^T Ax = \mathbf{A^T b}$. Here A^T is the **transpose** matrix, whose rows are **the columns of A**. The nullspace of A^T contains all solutions to $\mathbf{A^T y = 0}$. The **column** space of A^T (row space of A) is the fourth fundamental subspace. Each subspace has a basis containing as many **independent** vectors as possible. The number of vectors in the basis is the **dimension** of the subspace.

When $Ax = \lambda x$ the number λ is an **eigenvalue** and x is an **eigenvector**. The equation $dy/dt = Ay$ has the exponential solution $y = e^{At}y_0$. A 7 by 7 matrix has **seven** eigenvalues, whose product is the **determinant** D. If D is nonzero the matrix A has an **inverse**. Then $Ax = b$ is solved by $x = \mathbf{A^{-1}b}$. The formula for D contains $7! = 5040$ terms, so x is better computed by **elimination**. On the other hand $Ax = \lambda x$ means that $A - \lambda I$ has determinant **zero**. The eigenvalue is computed before the **eigenvector**.

1 All vectors $c\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **3** Only $x = 0$ **5** Plane of vectors with $x_1 + x_2 + x_3 = 0$

7 $x_p = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, A(x_p + x_0) = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **9** $A(x_p + x_0) = b + 0 = b$; another solution

11 $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}; b = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$

13 $CC^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}; C^TC = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$; (2 by 3) (2 by 3) is impossible

15 Any two are independent **17** C and F have independent columns

19 $\det F = 3$ **21** $F^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

23 $\det(F - \lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 3 - 4\lambda + \lambda^2 = 0$ if $\lambda = 1$ or $\lambda = 3$;

$F\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}, F\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

25 $y = e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}, y = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, y = \frac{e^t}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{e^{3t}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

27 $\det\begin{bmatrix}1-\lambda & 1 & 1\\ 1 & 1-\lambda & 1\\ 1 & 1 & 1-\lambda\end{bmatrix} = (1-\lambda)^3 - 3(1-\lambda) + 2 = \lambda^3 - 3\lambda^2 = 0$ if $\lambda = 3$ or $\lambda = 0$ (repeated)

29 $\det\begin{bmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{bmatrix} = \lambda^2 - 5\lambda = 0$ if $\lambda = 0$ or $\lambda = 5$; $A\begin{bmatrix}2\\-1\end{bmatrix} = 0\begin{bmatrix}2\\-1\end{bmatrix}$, $A\begin{bmatrix}1\\2\end{bmatrix} = 5\begin{bmatrix}1\\2\end{bmatrix}$

31 $H = \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$ **33** F if $b \neq 0$; T; T; F ($e^{\lambda t}$ is not a vector); T

2 $Bx = 0$ if $\mathbf{x} = \mathbf{c}\begin{bmatrix}1\\2\end{bmatrix}$ **4** $C^T x = 0$ if $\mathbf{x} = \mathbf{c}\begin{bmatrix}1\\2\\-1\end{bmatrix}$ **6** $Fx = 0$ only if $\mathbf{x} = \mathbf{0}$

8 $Ax = \begin{bmatrix}1\\0\end{bmatrix}$ has no solutions (parallel lines in the row picture)

10 If $Ax = b$ and $Ax_P = b$ then $\mathbf{A(x - x_P) = 0}$. The difference between any two solutions is a vector in the **nullspace**. (So any solution $\mathbf{x}$ equals $\mathbf{x}_{\text{particular}} + \mathbf{x}_{\text{homogeneous}}$.)

12 The row space of $A = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$ is the line of multiples of $\begin{bmatrix}1\\2\end{bmatrix}$. (So is the column space.)

The nullspace is the (perpendicular) line of multiples of $\begin{bmatrix}2\\-1\end{bmatrix}$.

14 $\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ has no solution because the first equations give $x_1 = 1$ and $x_2 = 1$, which violates the third equation. So multiply by C^T : $\begin{bmatrix}2 & 2\\ 2 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$ gives the best least squares solution $x_1 = \frac{2}{3}, x_2 = \frac{1}{3}$.

16 The vectors $\begin{bmatrix}1\\1\\3\end{bmatrix}$ and $\begin{bmatrix}1\\2\\5\end{bmatrix}$ are combinations of columns of C.

18 Ax is a combination of the columns. If $Ax = 0$ happens with a nonzero x, the columns are "dependent." If $Ax = 0$ only happens when $x = 0$, the columns are "independent."

20 $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$ are three independent vectors in 3D so they form a *basis*. The problem asks for three vectors with positive components not lying in a common plane.

22 $F^2 = \begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}$ has determinant $25 - 16 = 9$ which is $\mathbf{3^2} = (\det \mathbf{F})^2$.

24 $\det(G - \lambda I) = \begin{vmatrix}5-\lambda & 4\\ 4 & 5-\lambda\end{vmatrix} = \lambda^2 - 10\lambda + 9$. This is $(\lambda - 9)(\lambda - 1)$ which is zero at $\lambda = \mathbf{9}$ and $\lambda = \mathbf{1}$. Eigenvectors: $G\begin{bmatrix}1\\1\end{bmatrix} = 9\begin{bmatrix}1\\1\end{bmatrix}$ and $G\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$. These are also the eigenvectors of F. General reasoning: If $Fx = \lambda x$ then $F^2 x = \lambda F x = \lambda^2 x$. So x is also an eigenvector of F^2.

26 Exponential solutions to $\frac{dy}{dt} = Gy$ are $y_1 = e^{9t}\begin{bmatrix}1\\1\end{bmatrix}$ and $y_2 = e^t\begin{bmatrix}1\\-1\end{bmatrix}$. The combination $\frac{3}{2}y_1 + \frac{1}{2}y_2$ starts from $y_0 = \begin{bmatrix}3/2\\3/2\end{bmatrix} + \begin{bmatrix}1/2\\-1/2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$.

28 An eigenvector for $Ex = 0x$, when $\lambda = 0$, is the same as a solution to $Ex = 0$ (Problem 5).

$Ex = 3x$ is solved by $x = \begin{bmatrix}1\\1\\1\end{bmatrix}$ or any multiple $\begin{bmatrix}c\\c\\c\end{bmatrix}$.

30 A zero eigenvalue means there is a solution to $Ax = 0x = 0$. In this case A is not invertible and its determinant is zero. The reasoning also goes in reverse.

32 If $Fx = \lambda x$ then multiplying both sides by F^{-1} and λ^{-1} gives $\lambda^{-1}x = F^{-1}x$. If F has eigenvalues $\lambda = 1$ and 3, then F^{-1} has eigenvalues $\lambda^{-1} = 1$ and $\frac{1}{3}$ (with the same two eigenvectors x). The determinant of F is 1 times 3 and det F^{-1} is 1 times $\frac{1}{3}$.

16.2 Differential Equations (page 610)

The solution to $y' - 5y = 10$ is $y = Ae^{5t} + B$. The homogeneous part Ae^{5t} satisfies $y' - 5y = \mathbf{0}$. The particular solution B equals $\mathbf{-2}$. The initial condition y_0 is matched by $A = \mathbf{y_0 + 2}$. For $y' - 5y = e^{kt}$ the right form is $y = Ae^{5t} + \mathbf{Be^{kt}}$. For $y' - 5y = \cos t$ the form is $y = Ae^{5t} + \mathbf{a\cos t + b\sin t}$.

The equation $y'' + 4y' + 5y = 0$ is second-order because **it begins with y''**. The pure exponential solutions come from the roots of $\lambda^2 + 4\lambda + 5 = 0$, which are $r = -1$ and $s = -4$. The general solution is $y = \mathbf{Ae^{-t} + Be^{-4t}}$. Changing $4y'$ to **zero** yields pure oscillation. Changing to $2y'$ yields $\lambda = -1 \pm 2i$, when the solutions become $y = \mathbf{Ae^{-t}\cos 2t + Be^{-t}\sin 2t}$. This oscillation is underdamped. A spring with $m = 1, d = 2, k = 5$ goes directly to zero. An electrical network with $L = 1, R = 2, C = \frac{1}{5}$ also **goes to zero (underdamped)**.

One particular solution of $y'' + 4y = e^t$ is e^t times $\mathbf{\frac{1}{5}}$. If the right side is $\cos t$, the form of y_P is $\mathbf{\frac{1}{3}\cos t}$. If the right side is 1 then $y_P = \frac{1}{4}$. If the right side is **$\cos 2t$** we have resonance and y_P contains an extra factor **t**.

1 $3Be^{3t} - Be^{3t} = 8e^{3t}$ gives $B = 4 : y = 4e^{3t}$ **3** $y = 3 - 2t + t^2$ **5** $Ae^t + 4e^{3t} = 7$ at $t = 0$ if $A = 3$

7 Add $y = Ae^{-t}$ because $y' + y = 0$; choose $A = -1$ so $-e^{-t} + 3 - 2t + t^2 = 2$ at $t = 0$

9 $y = \frac{e^{kt}-1}{k}; y = t$; by l'Hôpital $\lim_{k\to 0} \frac{e^{kt}-1}{k} = \lim_{k\to 0} \frac{te^{kt}}{1} = t$

11 Substitute $y = Ae^t + Bte^t + C\cos t + D\sin t$ in equation: $B = 1, C = \frac{1}{2}, D = -\frac{1}{2}$, any A

13 Particular solution $y = Ate^t + Be^t; y' = Ate^t + (A+B)e^t = c(Ate^t + Be^t) + te^t$ gives $A = cA + 1, A + B = cB, A = \frac{1}{1-c}, B = \frac{-1}{(1-c)^2}$

15 $\lambda^2 e^{\lambda t} + 6\lambda e^{\lambda t} + 5e^{\lambda t} = 0$ gives $\lambda^2 + 6\lambda + 5 = 0, (\lambda+5)(\lambda+1) = 0, \lambda = -1$ or -5 (both negative so decay); $y = Ae^{-t} + Be^{-5t}$

17 $(\lambda^2 + 2\lambda + 3)e^{\lambda t} = 0, \lambda = -1 \pm \sqrt{-2}$ has imaginary part and negative real part; $y = Ae^{(-1+\sqrt{2}i)t} + Be^{(-1-\sqrt{2}i)t}; y = Ce^{-t}\cos\sqrt{2}t + De^{-t}\sin\sqrt{2}t$

19 $d = 0$ no damping; $d = 1$ underdamping; $d = 2$ critical damping; $d = 3$ overdamping

21 $\lambda = -\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$ is repeated when $b^2 = 4c$ and $\lambda = -\frac{b}{2}; (t\lambda^2 + 2\lambda)e^{\lambda t} + b(t\lambda + 1)e^{\lambda t} + cte^{\lambda t} = 0$ when $\lambda^2 + b\lambda + c = 0$ and $2\lambda + b = 0$

23 $-a\cos t - b\sin t - a\sin t + b\cos t + a\cos t + b\sin t = \cos t$ if $a = 0, b = 1, y = \sin t$

25 $y = A\cos 3t + B\cos 5t; y'' + 9y = -25B\cos 5t + 9B\cos 5t = \cos 5t$ gives $B = \frac{-1}{16}$; $y_0 = 0$ gives $A = \frac{1}{16}$

27 $y = A(\cos\omega t - \cos\omega_0 t), y'' = -A\omega^2\cos\omega t + A\omega_0^2\cos\omega_0 t, y'' + \omega_0^2 y = \cos\omega t$ gives $A(-\omega^2 + \omega_0^2) = 1$; breaks down when $\omega^2 = \omega_0^2$

29 $y = Be^{5t}; 25B + 3B = 1, B = \frac{1}{28}$ **31** $y = A + Bt = \frac{1}{2} + \frac{1}{2}t$

33 $y'' - 25y = e^{5t}; y'' + y = \sin t; y'' = 1 + t$; right side solves homogeneous equation so particular solution needs extra factor t

35 $e^t, e^{-t}, e^{it}, e^{-it}$ **37** $y = e^{-2t} + 2te^{-2t}; y(2\pi) = (1+4\pi)e^{-4\pi} \approx 0$

39 $y = (4e^{-rt} - r^2 e^{-4t/r})/(4-r^2) \to 1$ as $r \to 0$ **43** $h \le 2; h \le 2.8$

2 $-2a\sin 2t + 2b\cos 2t + a\cos 2t + b\sin 2t = 4\sin 2t$; collect $\sin 2t$ and $\cos 2t$ terms to find $-2a + b = 4$ and $2b + a = 0$; then $b = \frac{4}{5}, a = -\frac{8}{5}$, $\mathbf{y = -\frac{8}{5}\cos 2t + \frac{4}{5}\sin 2t}$.

4 $ae^t\cos t - ae^t\sin t + be^t\sin t + be^t cost = 2e^t\cos t$; collect terms to find $a + b = 2, -a + b = 0$; then $a = 1, b = 1 \mathbf{y = e^t(\cos t + \sin t)}$. This integral of $2e^t\cos t$ could be found by parts; here the correct form was assumed at the start.

6 Ae^{-t} solves the homogeneous equation $y' + y = 0$; then with $A = \frac{8}{5}$ the combination $\frac{8}{5} - \frac{8}{5}\cos 2t + \frac{4}{5}\sin 2t$ solves $y' + y = 4\sin 2t$ with $y(0) = 0$.

8 $y =$ constant solves the homogeneous equation $y' = 0$, so a constant can be added to any particular solution of $y' = 2e^t\cos t$ (the constant A is the same as the integration constant C!).

10 Substitute homogeneous solution plus particular solution $y = Ae^t + Be^{kt}$ into the equation; then $kB - B = 1$ and also $A + B = 0$ from $y_0 = 0$; thus $B = -A = \frac{1}{k-1}$ and $\mathbf{y = -\frac{e^t}{k-1} + \frac{e^{kt}}{k-1}}$. Apply l'Hôpital's rule (the variable is k!) to find $\lim_{k\to 1} \frac{-e^t + e^{kt}}{k-1} = \lim_{k\to 1} \frac{te^{kt}}{1} = te^t$.

12 Homogeneous solution plus particular solution $= Ae^{-t} + Be^t + Ct + D$. Substitute to find $2Be^t + C + Ct + D = e^t + t$ and $B = \frac{1}{2}, C = 1, D = -1$; then $\mathbf{y = Ae^{-t} + \frac{1}{2}e^t + t - 1}$. (Initial value y_0 determines A.)

14 Particular solution is $B + Dt$; substitute to find $D = B + Dt + t$; then $B = -1, D = -1$; general solution $\mathbf{y = Ae^t - t - 1}$.

16 Substitution gives $(\lambda^2 + 9)e^{\lambda t} = 0$. (a) $\lambda = 3i$ or $\lambda = -3i$ (b) Pure oscillation because the λ's are pure imaginary (c) General solution $y = Ae^{3it} + Be^{-3it} = \mathbf{a\cos 3t + b\sin 3t}$.

18 Substitution gives $(\lambda^2 + 6\lambda + 9)e^{\lambda t} = 0$. (a) $\lambda^2 + 6\lambda + 9 = (\lambda+3)^2$ and the equation $(\lambda+3)^2 = 0$ has the repeated solution $\lambda = -3$ (b) The general solution is $\mathbf{y = Ae^{-3t} + Bte^{-3t}}$.

20 $k = 0$ gives overdamping; $k = 1$ gives critical damping; $k = 2$ gives underdamping.

22 Substitute $(te^{-t})'' + 3(te^{-t})' + 2(te^{-t}) = (t-2)e^{-t} + 3(1-t)e^{-t} + 2te^{-t} = e^{-t} \neq$ zero. So te^{-t} is *not* a homogeneous solution. (It would be if the coefficients in the equation changed from 1,3,2 to 1,2,1; then $\lambda^2 + 2\lambda + 1$ has $\lambda = -1$ as a double root.)

24 Substitute y to find $-\omega^2 a\cos\omega t - \omega^2 b\sin\omega t - \omega a\sin\omega t + \omega b\cos\omega t + a\cos\omega t + b\sin\omega t = \sin\omega t$. Then $(1-\omega^2)a + \omega b = 0$ and $-\omega a + (1-\omega^2)b = 1$, which gives $\mathbf{a = \frac{-\omega}{\omega^2 + (1-\omega^2)^2}}$ and $\mathbf{b = \frac{1-\omega^2}{\omega^2 + (1-\omega^2)^2}}$.

26 The graph of $\cos 5t - \cos 3t$ will look like the figure following the exercises in Section 7.2, which shows $\sin 10x \sin x = \frac{1}{2}(\cos 11x - \cos 9x)$.

28 Substitute to find $(i\omega L + R + \frac{1}{i\omega C})Ae^{i\omega t} = Ve^{i\omega t}$. Then $A = \frac{V}{i\omega L + R + (i\omega C)^{-1}}$.

30 Substitute $y_{\text{particular}} = A\sin t + B\cos t$ to find $-A\sin t - B\cos t + 3A\sin t + 3B\cos t = \sin t$. Then $2A = 1$ and $B = 0$ give $\mathbf{y = \frac{1}{2}\sin t}$. (Note: the homogeneous solution involves $\cos\sqrt{3}t$ and $\sin\sqrt{3}t$.)

32 Substitute $y_{\text{particular}} = Ae^t\cos t + Be^t\sin t$ to find $-2Ae^t\sin t + 2Be^t\cos t + 2Ae^t\cos t + 2Be^t\sin t = e^t\cos t$. Then $-2A + 2B = 0$ and $2B + 2A = 1$ and $\mathbf{A = B = \frac{1}{4}}$.

34 Equation (2) used l'Hôpital's Rule for the limit of $y = \frac{e^{kt} - e^{ct}}{k-c}$ as k approached c. This exercise uses the series $e^{kt} = 1 + kt + \frac{1}{2}k^2t^2 + \cdots$ and $e^{ct} = 1 + ct + \frac{1}{2}c^2t^2 + \cdots$ to find $\frac{e^{kt} - e^{ct}}{k-c} = \frac{kt - ct + \frac{1}{2}k^2t^2 - \frac{1}{2}c^2t^2 + \cdots}{k-c} = t + \frac{1}{2}(k+c)t^2 + \cdots$. As k approaches c this is $\mathbf{t + ct^2} + \cdots = t(1 + ct + \cdots) = \mathbf{te^{ct}}$.

36 Substitute $y_{\text{particular}} = Ae^t$ to find $Ae^t + Ae^t = e^t$. Then $A = \frac{1}{2}$ and $\mathbf{y = \frac{1}{2}e^t}$.

38 $y_0 = 1$ gives $A + B = 1$; $y_0' = 0$ gives $-A - 4B = 0$; the solution has $A = \frac{4}{3}, B = -\frac{1}{3}$, and $y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$. At $t = 2\pi$ the value is $\mathbf{\frac{4}{3}e^{-2\pi} - \frac{1}{3}e^{-8\pi}} \approx \frac{4}{3}e^{-2\pi}$.

40 In exponential solutions Ae^{kt} all derivatives are proportional to the function. For $y'' = 6y^2$ substitute $y = x^n$ to find $n(n-1)x^{n-2} = 6x^{2n}$. Comparing exponents gives $n - 2 = 2n$ or $n = -2$. Then $n(n-1) = 6$ and a solution is $\mathbf{y = x^{-2}}$. (If the 6 were changed, this solution would become Ax^{-2}.)

44 Runge-Kutta is stable with $h = .02$ but 100 steps with $h = .03$ lead to $y_{100} \approx 670,000,000,000$.

16.3 Discrete Mathematics (page 615)

A graph is a set V of **nodes or vertices** and a set E of **edges**. With 6 nodes, a complete graph has **15** edges. A spanning tree has only **5 edges**. A tree is defined as **a graph with no loops**, and it is spanning if it **contains all nodes**. It has one path between each pair of nodes.

To find a path from node i to node j, two search methods are **depth first search and breadth first search**. As nodes are reached, DFS looks out from the **latest** node for a new one. BFS looks out from the **earliest node**. DFS must be prepared to **backtrack** to earlier nodes. In case of fire, BFS locates all doors from the room you are in before it **chooses one**.

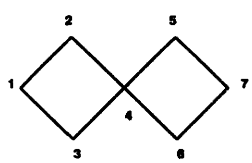
1 Two then two then last one; go around hexagon **3** Six (each deletes one edge)

5 Connected: there is a path between any two nodes; connecting each new node requires an edge

13 Edge lengths 1,2,4

15 No; 1,3,4 on left connect only to 2,3 on right; 1,3 on right connect only to 2 on left **17** 4

19 Yes **21** F (may loop); T **25** 16

2 

Breadth first search from node 4: 2, 3, 5, 6, 1, 7.
Depth first search from node 4: 2, 1, 3, (backtrack to 4), 5, 7, 6

4 A spanning tree omits one edge from each square. There are 4×4 choices of edges to omit: 16 spanning trees.

6 A loop is a sequence of edges from x_0 to x_1, x_1 to $x_2, \cdots, x_k$ to x_0 : the loop (or circuit) is a closed path on the graph. A loop involving k nodes uses k edges. That leaves only $8 - k$ edges to connect the loop to the remaining $9 - k$ nodes, which is impossible.

8 Breadth first search finds all neighbors, then all neighbors of neighbors, and so on. The farthest state is found last.

10 Please send answer!

12 A matching is a set of edges, no two of which touch the same node.

14 The shortest path tree in network A contains the edges of length 4, 5, 1.

16 The loop in network B contains the edges of length 1, 4, 2, 3. Minimum spanning tree by Method 1, starting top left: 7, 2, 3, 1, 5, 6, 8. Minimum spanning tree by Method 2 (increasing lengths, avoiding loops): 1, 2, 3, (avoid 4), 5, 6, 7, 8.

18 If one node has 4 edges going out, then every other node has one of these edges coming in: a node with 0 edges cannot exist.

20 Suppose a spanning tree contains edge (8) but not (6). Then it is shorter if we substitute edge (6) for edge (8). Also *it is still a spanning tree:* It contains the correct number of edges and no loops. Edge (6) cannot be in a loop since edge (8) was removed; and those are the only edges into their common node.

22 (a) Starting from node s, a tree that connects each node to the next is perfect for depth first search. There are no backtracks. (b) For breadth first search the perfect tree connects node s directly to each other node. On that tree, depth first search would have to backtrack after each new node.

24 The code intends to find shortest distances d_{ij} between every pair of nodes (like a road atlas). Correction: *The loop on k should be the outer loop.* Then $k = 1$ allows direct edges i to j and paths from i to 1 to j. Next $k = 2$ also allows paths from i to 2 to j and from i to 2 to 1 to j and from i to 1 to 2 to j. Eventually all intermediate nodes and all paths are allowed.

26 Maximum spanning tree: Method 1 adds the longest edge that goes out from the current tree (start at any node). Method 2 adds edges in decreasing order, longest first, rejecting any edge that closes a loop.