CHAPTER 13 PARTIAL DERIVATIVES

13.1 Surfaces and Level Curves (page 475)

The graph of $z = f(x,y)$ is a **surface in three-dimensional space.** The level curve $f(x,y) = 7$ lies down in the base plane. Above this level curve are all points at height **7** in the surface. The **plane** $z = 7$ cuts through the surface at those points. The level curves $f(x,y) = c$ are drawn in the xy plane and labeled by **c**. The family of labeled curves is a **contour** map.

For $z = f(x,y) = x^2 - y^2$, the equation for a level curve is $\mathbf{x^2 - y^2 = c}$. This curve is a **hyperbola.** For $z = x - y$ the curves are **straight lines.** *Level curves never cross because* **f(x,y) cannot equal two numbers c and c′.** They crowd together when the surface is **steep.** The curves tighten to a point when **f reaches a maximum or minimum.** The steepest direction on a mountain is **perpendicular** to the level curve.

3 x derivatives $\infty, -1, -2, -4e^{-4}$ (flattest) **5** Straight lines **7** Logarithm curves

9 Parabolas **11** No: $f = (x+y)^n$ or $(ax+by)^n$ or any function of $ax+by$ **13** $f(x,y) = 1 - x^2 - y^2$

15 Saddle **17** Ellipses $4x^2 + y^2 = c^2$ **19** Ellipses $5x^2 + y^2 = c^2 + 4cx + x^2$

21 Straight lines not reaching (1,2) **23** Center $(1,1)$; $f = x^2 + y^2 - 1$ **25** Four, three, planes, spheres

27 Less than 1, equal to 1, greater than 1 **29** Parallel lines, hyperbolas, parabolas

31 $\frac{d}{dx}: 48x - 3x^2 = 0, x = 16$ hours **33** Plane; planes; 4 left and 3 right (3 pairs)

2 Level curves are **circles** for any function of $x^2 + y^2$; the maximum is at $(0,0)$; the functions equal 1 when $x^2 + y^2 = 3, 1, 2, \infty$ (radius is square root: increasing order $\mathbf{f_2, f_3, f_1, f_4}$).

4 $\frac{d}{dx}\sqrt{3 - x^2} = \frac{-x}{\sqrt{3-x^2}} = \frac{-1}{\sqrt{2}}$ at $x = 1$; $\frac{df_2}{dx} = \frac{-x}{\sqrt{x^2+1}} = \frac{-1}{\sqrt{2}}$; $\frac{df_3}{dx} = -x = -1$; $\frac{df_4}{dx} = -2xe^{-x^2-1} = -2e^{-2}$.

6 $(x+y)^2 = 0$ gives the **line** $\mathbf{y = -x}$; $(x+y)^2 = 1$ gives the **pair of lines** $x + y = 1$ and $x + y = -1$; similarly $\mathbf{x + y = \sqrt{2}}$ and $\mathbf{x + y = -\sqrt{2}}$; **no level curve** $(x+y)^2 = -4$.

8 $\sin(x-y) = 0$ on an infinite set of **parallel lines** $x - y = 0, \pm\pi, \pm 2\pi, \cdots$; for $c = 1$ the level curves $\sin(x-y) = 1$ are **parallel lines** $x - y = \frac{\pi}{2} + 2\pi n$; **no level curves** for $c = 2$ and $c = -4$.

10 The curve $\frac{y}{x^2} = 0$ is the **axis** $\mathbf{y = 0}$ excluding (0,0); $\frac{y}{x^2} = 1$ or 2 or -4 is a **parabola.**

12 $f(x,y) = xy - 1$ has level curve $f = 0$ as two pieces of a hyperbola.

14 $f(x,y) = \sin(x+y)$ is zero on infinitely many lines $x + y = 0, \pm\pi, \pm 2\pi, \cdots$

16 $f(x,y) = \{$ maximum of $x^2 + y^2 - 1$ and zero $\}$ is zero inside the unit circle.

18 $\sqrt{4x^2 + y^2} = c + 2x$ gives $4x^2 + y^2 = c^2 + 4cx + 4x^2$ or $\mathbf{y^2 = c^2 + 4cx}$. This is a **parabola** opening to the left or right.

20 $\sqrt{3x^2 + y^2} = c + 2x$ gives $3x^2 + y^2 = c^2 + 4cx + 4x^2$ or $y^2 - x^2 = c^2 + 4cx$. This is a **hyperbola.**

26 Since $x^2 + y^2$ is always ≥ 0, the surface $x^2 + y^2 = z^2 - 1$ has no points with z^2 less than 1.

30 Direct approach: $xy = (\frac{x_1+x_2}{2})(\frac{y_1+y_2}{2}) = \frac{1}{4}(x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1) = \frac{1}{4}(1 + 1 + \frac{x_1}{x_2} + \frac{x_2}{x_1})$ $= 1 + \frac{(x_1-x_2)^2}{4x_1x_2} \geq 1$. **Quicker approach:** $y = \frac{1}{x}$ is concave up (or convex) because $y'' = \frac{2}{x^3}$ is positive.

Note for convex functions: Tangent lines below curve, secant line segments above curve!

32 $y = \frac{2048}{x^2}$ has $\frac{dy}{dx} = -\frac{4096}{x^3} = \mathbf{-1}$ at $x = 16$. Also $y'' = \frac{12{,}288}{x^4} \geq 0$ so the curve is concave up (or convex). The line $x + y = 24$ also goes through (16,8) with slope $\mathbf{-1}$; it must be the tangent line.

34 The function $f(x, y)$ is the height above the ground. The level curve $f = 0$ is the outline of the shoe.

13.2 Partial Derivatives (page 479)

The **partial** derivative $\partial f/\partial y$ comes from fixing $\mathbf{x}$ and moving $\mathbf{y}$. It is the limit of $\mathbf{f(x, y + \Delta y) - f(x, y))/\Delta y}$. If $f = e^{2x} \sin y$ then $\partial f/\partial x = \mathbf{2e^{2x} \sin y}$ and $\partial f/\partial y = \mathbf{e^{2x} \cos y}$. If $f = (x^2+y^2)^{1/2}$ then $f_x = \mathbf{x/(x^2 + y^2)^{1/2}}$ and $f_y = \mathbf{y/(x^2 + y^2)^{1/2}}$. At (x_0, y_0) the partial derivative f_x is the ordinary derivative of the **partial** function $f(x, y_0)$. Similarly f_y comes from $\mathbf{f(x_0, y)}$. Those functions are cut out by vertical planes $x = x_0$ and $\mathbf{y = y_0}$, while the level curves are cut out by **horizontal** planes.

The four second derivatives are $f_{xx}, \mathbf{f_{xy}, f_{yx}, f_{yy}}$. For $f = xy$ they are $\mathbf{0, 1, 1, 0}$. For $f = \cos 2x \cos 3y$ they are $\mathbf{-4 \cos 2x \cos 3y, 6 \sin 2x \sin 3y, -9 \cos 2x \cos 3y}$. In those examples the derivatives $\mathbf{f_{xy}}$ and $\mathbf{f_{yx}}$ are the same. That is always true when the second derivatives are **continuous**. At the origin, $\cos 2x \cos 3y$ is curving **down** in the x and y directions, while xy goes **up** in the 45° direction and **down** in the −45° direction.

1 $3 + 2xy^2; -1 + 2yx^2$ **3** $3x^2y^2 - 2x; 2x^3y - e^y$ **5** $\frac{-2y}{(x-y)^2}; \frac{2x}{(x-y)^2}$ **7** $\frac{-2x}{(x^2+y^2)^2}; \frac{-2y}{(x^2+y^2)^2}$
9 $\frac{x}{x^2+y^2}; \frac{y}{x^2+y^2}$ **11** $\frac{-y}{x^2+y^2}; \frac{x}{x^2+y^2}$ **13** $2, 3, 4$ **15** $6(x + iy), 6i(x + iy), -6(x + iy)$
17 $(f = \frac{1}{r}) f_{xx} = \frac{2x^2-y^2}{r^5}; f_{xy} = \frac{3xy}{r^5}; f_{yy} = \frac{2y^2-x^2}{r^5}$ **19** $-a^2 \cos ax \cos by, ab \sin ax \sin by, -b^2 \cos ax \cos by$
21 Omit line $x = y$; all positive numbers; $f_x = -2(x - y)^{-3}, f_y = 2(x - y)^{-3}$
23 Omit $z = t$; all numbers; $\frac{-1}{z-t}, \frac{1}{z-t}, \frac{(x-y)}{(z-t)^2}, \frac{(y-x)}{(z-t)^2}$
25 $x > 0, t > 0$ and $x = 0, t > 1$ and $x = -1, -2, \cdots, t = e, e^2, \cdots; f_x = (\ln t)x^{\ln t - 1}, f_t = (\ln x)t^{\ln x - 1}$
27 $y, x; f = G(x) + H(y)$ **29** $\frac{\partial f}{\partial x} = \frac{\partial (xy)}{\partial x} v(xy) = yv(xy)$
31 $f_{xxx} = 6y^3, f_{yyy} = 6x^3, f_{xxy} = f_{xyx} = f_{yxx} = 18xy^2, f_{yyx} = f_{yxy} = f_{xyy} = 18x^2y$
33 $g(y) = Ae^{cy/7}$ **35** $g(y) = Ae^{cy/2} + Be^{-cy/2}$
37 $f_t = -2f, f_{xx} = f_{yy} = -e^{-2t} \sin x \sin y; e^{-13t} \sin 2x \sin 3y$
39 $\sin(x + t)$ moves left **41** $\sin(x - ct), \cos(x + ct), e^{x-ct}$
43 $(B - A)h_y(C^*) = (B - A)[f_y(b, C^*) - f_y(a, C^*)] = (B - A)(b - a)f_{yx}(c^*, C^*)$; continuous f_{xy} and f_{yx}
45 y converges to b; inside and stay inside; $d_n = \sqrt{(x_n - a)^2 + (y_n - b)^2} \to$ zero; $d_n < \epsilon$ for $n > N$
47 ϵ, less than δ **49** $f(a, b); \frac{1}{x-1}$ or $\frac{1}{(x-1)(y-2)}$ **51** $f(0, 0) = 1; f(0, 0) = 1$; not defined for $x < 0$

2 $\frac{\partial f}{\partial x} = 3 \cos(3x - y), \frac{\partial f}{\partial y} = -\cos(3x - y) + 1$ **4** $\frac{\partial f}{\partial x} = e^{x+4} + xe^{x+4}, \frac{\partial f}{\partial y} = 0$
6 $\frac{\partial f}{\partial x} = -x(x^2 + y^2)^{-3/2}, \frac{\partial f}{\partial y} = -y(x^2 + y^2)^{-3/2}$ **8** $\frac{\partial f}{\partial x} = \frac{1}{x+2y}, \frac{\partial f}{\partial y} = \frac{2}{x+2y}$ **10** $\frac{\partial f}{\partial x} = y^x(\ln y), \frac{\partial f}{\partial y} = xy^{x-1}$
12 $\frac{\partial f}{\partial x} = \frac{1}{x}, \frac{\partial f}{\partial y} = \frac{1}{y}$ **14** $f_{xx} = 2, f_{xy} = f_{yx} = 6, f_{yy} = 18$
16 $f_{xx} = a^2e^{ax+by}, f_{xy} = f_{yx} = abe^{ax+by}, f_{yy} = b^2e^{ax+by}$
18 $f_{xx} = n(n - 1)(x + y)^{n-2} = f_{xy} = f_{yx} = f_{yy}$!
20 $f_{xx} = \frac{2}{(x+iy)^3}, f_{xy} = f_{yx} = \frac{2i}{(x+iy)^3}, f_{yy} = \frac{2i^2}{(x+iy)^3} = \frac{-2}{(x+iy)^3}$ Note $f_{xx} + f_{yy} = 0$.
22 Domain: all (x, y, t) such that $x^2 + y^2 \geq t^2$ (interior of cone $x^2 + y^2 = t^2$); range: all values $f \geq 0$;

$\frac{\partial f}{\partial x} = \frac{x}{f}, \frac{\partial f}{\partial y} = \frac{y}{f}, \frac{\partial f}{\partial t} = -\frac{t}{f}$.

24 Domain: halfplane where $x + t > 0$; range: all real numbers; $f_x = \frac{1}{x+t} = f_t$.

26 Domain: all (x, y) with $|y| \leq 1$; range: all numbers with absolute value $|f(x, y)| \leq \pi$ (since $-1 \leq \cos\ x \leq 1$ and $0 \leq \cos^{-1} y \leq \pi$); $\frac{\partial f}{\partial x} = -\sin\ x \cos^{-1} y$, $\frac{\partial f}{\partial y} = -\frac{\cos\ x}{\sqrt{1-y^2}}$.

28 $\frac{\partial f}{\partial x} = -v(x)$ and $\frac{\partial f}{\partial y} = v(y)$.

30 $f(x, y) = \int_x^y \frac{dt}{t} = \ln y - \ln x = \ln \frac{y}{x}$; $f_x = -\frac{1}{x}$ and $f_y = \frac{1}{y}$ (confirming Problem 28).

32 $g(y) = e^{-cy}$ or any multiple Ae^{-cy}. **34** $g(y) = e^{c^2 y}$ or any multiple $Ae^{c^2 y}$.

36 $f_x = \frac{1}{\sqrt{t}}(\frac{-2x}{4t})e^{-x^2/4t}$. Then $f_{xx} = f_t = \frac{-1}{2t^{3/2}}e^{-x^2/4t} + \frac{x^2}{4t^{5/2}}e^{-x^2/4t}$.

38 $e^{-m^2t-n^2t} \sin mx\ \cos ny$ solves $f_t = f_{xx} + f_{yy}$. Also $f = \frac{1}{t}e^{-(x^2+y^2)/4t}$ has $f_t = f_{xx} + f_{yy} = (-\frac{1}{t^2} + \frac{x^2+y^2}{4t^3})e^{-(x^2+y^2)/4t}$.

40 $f = \sin(x - t)$ peaks when $x - t = \frac{\pi}{2}$ and $f = 1$. If t increases by $\Delta t = \frac{\pi}{4}$ then x increases by $\Delta x = \frac{\pi}{4}$. The wave velocity is $\frac{\Delta x}{\Delta t} = 1$. The other function $\sin(x + t)$ has $\frac{\Delta x}{\Delta t} = -1$ (velocity 1 to the *left*).

42 $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}$ gives $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial t \partial x}$ and also $\frac{\partial^2 f}{\partial x\ \partial t} = \frac{\partial^2 f}{\partial x^2}$. Use $f_{xt} = f_{tx}$ to find $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$.

44 (a) $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$ when $xy > 0$ (two quadrants); $\frac{\partial f}{\partial x} = -y$ and $\frac{\partial f}{\partial y} = -x$ when $xy < 0$ (other two quadrants); $\frac{\partial f}{\partial x}$ doesn't exist but $\frac{\partial f}{\partial y} = 0$ up the y axis; $\frac{\partial f}{\partial x} = 0$ but $\frac{\partial f}{\partial y}$ doesn't exist on the x axis.
(b) $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 2y$ except f is not continuous when $x = 0$ and $y \neq 0$. OK at (0,0).

46 (a) $(x_2, y_2) = (\frac{1}{3}, \frac{2}{3}); (x_4, y_4) = (\frac{1}{5}, \frac{4}{5})$; approaching **(0,1)** (b) $(x_2, y_2) = (x_4, y_4) = (1, 0)$ approaching **(1,0)** (c) $(x_2, y_2) = (x_4, y_4) = (1, 0)$ but **no limit** (d) $(x_2, y_2) = (2, 0), (x_4, y_4) = (4, 0)$ has **no limit**.

48 (a) The limit is $\sqrt{a^2 + b^2}$ (**continuous** function) (b) The limit is $\frac{a}{b}$ provided $\mathbf{b} \neq \mathbf{0}$ (c) The limit is $\frac{1}{a+b}$ provided $\mathbf{a + b} \neq \mathbf{0}$ (d) The limit is $\frac{ab}{a^2+b^2}$ except **no limit at (0,0)**.

50 Along $y = mx$ the function is $\frac{mx^3}{x^4+m^2x^2} \to 0$ (the ratio is near $\frac{mx^3}{m^2x^2}$ for small x). But on the parabola $y = x^2$ the function is $\frac{x^4}{2x^4} = \frac{1}{2}$. So this function $f(x, y)$ has **no limit**: not continuous at (0,0).

52 (a) $\frac{xy^2}{x^2+y^2} = y(\frac{xy}{x^2+y^2}) \to 0$ because always $|\frac{xy}{x^2+y^2}| \leq \frac{1}{2}$. (b) $\frac{x^2y^2}{x^4+y^4}$ equals 0 on the axes but $\frac{1}{2}$ on 45° lines; **no limit**; (c) $\frac{x^m y^n}{x^m+y^n} = (x^{m/2}y^{n/2})(\frac{x^{m/2}y^{n/2}}{x^m+y^n}) \to 0$ if $m > 0, n > 0$, because the second factor is $\leq \frac{1}{2}$ as in (a). For negative x and y, m and n should be positive integers. *Further problem* by same method: $\frac{x^a y^b}{x^m+y^n} \to 0$ if $a > \frac{m}{2}$ and $b > \frac{n}{2}$.

13.3 Tangent Planes and Linear Approximations (page 488)

The tangent line to $y = f(x)$ is $y - y_0 = \mathbf{f'(x_0)(x - x_0)}$. The tangent plane to $w = f(x, y)$ is $w - w_0 = \mathbf{(\partial f/\partial x)_0(x - x_0) + (\partial f/\partial y)_0(y - y_0)}$. The normal vector is $\mathbf{N} = \mathbf{(f_x, f_y, -1)}$. For $w = x^3 + y^3$ the tangent equation at (1,1,2) is $\mathbf{w - 2 = 3(x - 1) + 3(y - 1)}$. The normal vector is $\mathbf{N} = \mathbf{(3, 3, -1)}$. For a sphere, the direction of $\mathbf{N}$ is **out from the origin**.

The surface given implicitly by $F(x, y, z) = c$ has tangent plane with equation $(\partial F/\partial x)_0(x - x_0) + \mathbf{(\partial F/\partial y)(y - y_0) + (\partial F/\partial z)_0(z - z_0) = 0}$. For $xyz = 6$ at (1,2,3) the tangent plane has the equation $\mathbf{6(x - 1) + 3(y - 2) + 2(z - 3) = 0}$. On that plane the differentials satisfy $\mathbf{6}dx + \mathbf{3}dy + \mathbf{2}dz = 0$. The differential of $z = f(x, y)$ is $dz = \mathbf{f_x dx + f_y dy}$. This holds exactly on the tangent plane, while $\Delta z \approx \mathbf{f_x \Delta x + f_y \Delta y}$ holds approximately on the **surface**. The height $z = 3x + 7y$ is more sensitive to a change in $\mathbf{y}$ than in x, because the partial derivative $\partial \mathbf{z}/\partial \mathbf{y} = \mathbf{7}$ is larger than $\partial \mathbf{z}/\partial \mathbf{x} = \mathbf{3}$.

The linear approximation to $f(x,y)$ is $f(x_0,y_0)+(\partial \mathbf{f}/\partial \mathbf{x})_0(\mathbf{x}-\mathbf{x_0})+(\partial \mathbf{f}/\partial \mathbf{y})_0(\mathbf{y}-\mathbf{y_0})$. This is the same as $\Delta f \approx (\partial \mathbf{f}/\partial \mathbf{x})\Delta x + (\partial \mathbf{f}/\partial \mathbf{y})\Delta y$. The error is of order $(\boldsymbol{\Delta}\mathbf{x})^2 + (\boldsymbol{\Delta}\mathbf{y})^2$. For $f = \sin xy$ the linear approximation around (0,0) is $f_L = \mathbf{0}$. We are moving along the **tangent plane** instead of the **surface.** When the equation is given as $F(x,y,z)=c$, the linear approximation is $\mathbf{F_x}\Delta x + \mathbf{F_y}\Delta y + \mathbf{F_z}\Delta z = 0$.

Newton's method solves $g(x,y)=0$ and $h(x,y)=0$ by a **linear** approximation. Starting from x_n, y_n the equations are replaced by $\mathbf{g_x}\boldsymbol{\Delta}\mathbf{x} + \mathbf{g_y}\boldsymbol{\Delta}\mathbf{y} = -\mathbf{g}(\mathbf{x_n},\mathbf{y_n})$ **and** $\mathbf{h_x}\boldsymbol{\Delta}\mathbf{x} + \mathbf{h}y\boldsymbol{\Delta}\mathbf{y} = -\mathbf{h}(\mathbf{x_n},\mathbf{y_n})$. The steps Δx and Δy go to the next point $(\mathbf{x_{n+1}},\mathbf{y_{n+1}})$. Each solution has a basin of **attraction.** Those basins are likely to be **fractals.**

1 $z-1=y-1; \mathbf{N}=\mathbf{j}-\mathbf{k}$ **3** $z-2=\frac{1}{3}(x-6)-\frac{2}{3}(y-3); \mathbf{N}=\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\mathbf{k}$

5 $2(x-1)+4(y-2)+2(z-1)=0; \mathbf{N}=2\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ **7** $z-1=x-1; \mathbf{N}=\mathbf{i}-\mathbf{k}$

9 Tangent plane $2z_0(z-z_0)-2x_0(x-x_0)-2y_0(y-y_0)=0$; $(0,0,0)$ satisfies this equation because $z_0^2-x_0^2-y_0^2=0$ on the surface; $\cos\theta = \frac{\mathbf{N}\cdot\mathbf{k}}{|\mathbf{N}||\mathbf{k}|} = \frac{-z_0}{\sqrt{x_0^2+y_0^2+z_0^2}} = \frac{-1}{\sqrt{2}}$ (surface is the 45° cone)

11 $dz=3dx-2dy$ for both; $dz=0$ for both; $\Delta z=0$ for $3x-2y$, $\Delta z=.00029$ for x^3/y^2; tangent plane

13 $z=z_0+F_zt$; plane $6(x-4)+12(y-2)+8(z-3)=0$; normal line $x=4+6t, y=2+12t, z=3+8t$

15 Tangent plane $4(x-2)+2(y-1)+4(z-2)=0$; normal line $x=2+4t, y=1+2t, z=2+4t$; $(0,0,0)$ at $t=-\frac{1}{2}$

17 $dw=y_0dx+x_0dy$; product rule; $\Delta w-dw=(x-x_0)(y-y_0)$

19 $dI=4000dR+.08dP; dP=\$100; I=(.78)(4100)=\$319.80$

21 Increase $=\frac{26}{101}-\frac{25}{100}=\frac{3}{404}$, decrease $=\frac{25}{100}-\frac{25}{101}=\frac{1}{404}; dA=\frac{1}{y}dx-\frac{x}{y^2}dy$; 3 **23** $\Delta\theta \approx \frac{-y\Delta x+x\Delta y}{\sqrt{x^2+y^2}}$

25 Q increases; $Q_s=-\frac{250}{3}, Q_t=\frac{-5}{3}, P_s=-.2Q_s=\frac{50}{3}, P_t=-.2Q_t=\frac{1}{3}; Q=50-\frac{250}{3}(s-.4)-\frac{5}{3}(t-10)$

27 $s=1, t=10$ gives $Q=40$: $\begin{array}{rcl} P_s &=& -Q_s = sQ_s+Q=Q_s+40 \\ P_t &=& -Q_t = sQ_t+1=Q_t+1\end{array}$ $; Q_s=-20, Q_t=-\frac{1}{2}, P_s=20, P_t=\frac{1}{2}$

29 $z-2=x-2+2(y-1)$ and $z-3=4(x-2)-2(y-1); x=1, y=\frac{1}{2}, z=0$

31 $\Delta x=-\frac{1}{2}, \Delta y=\frac{1}{2}; x_1=\frac{1}{2}, y_1=-\frac{1}{2}$; line $x+y=0$

33 $\begin{array}{l} 3a^2\Delta x-\Delta y=-a-a^3 \\ -\Delta x+3a^2\Delta y=a+a^3\end{array}$ gives $\Delta y=-\Delta x=\frac{a+a^3}{1+3a^2}$; lemon starts at $(1/\sqrt{3},-1/\sqrt{3})$

35 If $x^3=y$ then $y^3=x^9$. Then $x^9=x$ only if $x=0$ or 1 or -1 (or complex number)

37 $\Delta x=-x_0+1, \Delta y=-y_0+2, (x_1,y_1)=(1,2)=$ solution

39 $G=H=\frac{x_n^2}{2x_n-1}$ **41** $J=\begin{bmatrix} e^x & 0 \\ 1 & e^y\end{bmatrix}, \Delta x=-1+e^{-x_n}, \Delta y=-1-(x_n-1+e^{-x_n})e^{-y_n}$

43 $(x_1,y_1)=(0,\frac{5}{4}),(-\frac{3}{4},\frac{5}{4}),(\frac{3}{4},0)$

2 $\mathbf{N}=\mathbf{i}+\mathbf{j}+\mathbf{k}; (x-3)+(y-4)+(z-10)=0$ **4** $\mathbf{N}=\mathbf{i}+2\mathbf{j}-\mathbf{k}; x+2y=z-1$

6 $\mathbf{N}=2\mathbf{i}+4\mathbf{j}+4\mathbf{k}; 2(x-1)+4(y-2)+4(z-1)=0$

8 $\mathbf{N}=8\pi\mathbf{i}+4\pi\mathbf{j}-\mathbf{k}; 8\pi(r-2)+4\pi(h-2)=V-8\pi$

10 $\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -1 \\ 2 & 3 & -1\end{vmatrix} = -\mathbf{i}-\mathbf{j}-5\mathbf{k}$ (both planes go through (0,0,0) and so does the line!)

12 $N_1 = 2i + 4j - k$ and $N_2 = 2i + 6j - k$ give $v = \begin{vmatrix} i & j & k \\ 2 & 4 & -1 \\ 2 & 6 & -1 \end{vmatrix} = 2i + 4k$ tangent to both surfaces

14 The direction of N is $2xy^2 i + 2x^2 y j - k = 8i + 4j - k$. So the line through (1,2,4) has $x = \mathbf{1 + 8t}, y = \mathbf{2 + 4t}$, $z = \mathbf{4 - t}$.

16 The normal line through (x_0, y_0, z_0) has direction $N = (F_x i + F_y j + F_z k)_0$. This is the radial line from the origin if $(F_x)_0 = cx_0, (F_y)_0 = cy_0, (F_z)_0 = cz_0$. Then F is a function of $x^2 + y^2 + z^2$ and the surface is a sphere.

18 $df = yz\,dx + xz\,dy + xy\,dz$.

20 Direct method: $R = \frac{R_1 R_2}{R_1 + R_2}$ and $\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1+R_2)^2}$ and $\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1+R_2)^2}$. If $R_1 = 1$ and $R_2 = 2$ then $\frac{\partial R}{\partial R_1}$ is four times larger ($\frac{4}{9}$ vs. $\frac{1}{9}$; **more sensitive to $\mathbf{R_1}$**). By chain rule: $-\frac{1}{R^2}\frac{\partial R}{\partial R_1} = -\frac{1}{R_1^2}$ and $-\frac{1}{R_2^2}\frac{\partial R}{\partial R_2} = -\frac{1}{R_2^2}$.

22 (a) Common sense: 2 hits in 5 at bats ($\frac{x}{y} = .4$) raises an average that is **below** .4. Mathematics: $A = \frac{x}{y}$ has $dA = \frac{y\,dx - x\,dy}{y^2} > 0$ if $y\,dx > x\,dy$. This is again $\frac{dx}{dy} > \frac{x}{y}$ or $.4 > A$. (b) The player has $x = 200$ hits since $\frac{200}{400} = .5$. We want to choose $\Delta x = \Delta y$ (all hits) so ΔA reaches .005. But $\Delta A \approx \frac{y\Delta x - x\Delta y}{y^2} = \frac{200}{(400)^2}\Delta x = .005$ when $\Delta x = \mathbf{4}$ **hits**. Check: $\frac{204}{404} = .505$ (to 3 decimals). If averages are rounded down we need $\Delta x = 5$ hits.

24 (1) c **is between $\mathbf{x_0}$ and $\mathbf{x_0 + \Delta x}$** by the Mean Value Theorem (2) C **is between $\mathbf{y_0}$ and $\mathbf{y_0 + \Delta y}$** (3) the limit exists if **$\mathbf{f_x}$ is continuous** (4) the limit exists if **$\mathbf{f_y}$ is continuous.**

26 $P = \frac{40s + .2t}{s + .2}$ and $Q = \frac{40 - t}{s + .2}$ so $\frac{\partial P}{\partial s} = \frac{8 - .2t}{(s+.2)^2}$ and $\frac{\partial P}{\partial t} = \frac{.2}{s + .2}$. At $s = .4, t = 10$ this gives $\frac{\partial P}{\partial s} = \frac{6}{(.6)^2} = \frac{50}{3}$ and $\frac{\partial P}{\partial t} = \frac{.2}{.6} = \frac{1}{3}$.

28 Take partial derivatives with respect to b: $2x\frac{\partial x}{\partial b} + b\frac{\partial x}{\partial b} + x = 0$ or $\frac{\partial x}{\partial b} = \mathbf{\frac{-x}{2x + b}}$. Similarly $2x\frac{\partial x}{\partial c} + b\frac{\partial x}{\partial c} + 1 = 0$ gives $\frac{\partial x}{\partial c} = \mathbf{\frac{-1}{2x + b}}$. Then $\frac{\partial x}{\partial b}$ is larger (in magnitude) when $x = 2$.

30 (a) The third surface is **z=0**. (b) Newton uses the tangent plane to the graph of g, **the tangent plane to the graph of h, and z = 0.**

32 $\frac{3}{4}\Delta x - \Delta y = \frac{3}{8}$ and $-\Delta x + \frac{3}{4}\Delta y = \frac{3}{8}$ give $\Delta x = \Delta y = -\frac{3}{2}$. The new point is $(-1, -1)$, an exact solution. The point $(\frac{1}{2}, \frac{1}{2})$ is in the gray band (upper right in Figure 13.11a) or the blue band on the front cover of the book.

34 $3a^2\Delta x - \Delta y = -a^3$ and $-\Delta x + 0\Delta y = a$ give $\Delta x = -a$ and $\Delta y = -2a^3$. The new point is $(0, -2a^3)$ on the y axis. Then $0\Delta x - \Delta y = -2a^3$ and $-\Delta x + 3(4a^6)\Delta y = 8a^9$ give $\Delta y = 2a^3$ and $\Delta x = 16a^9$. The new point $(16a^9, 0)$ is the same as the start $(a, 0)$ if $16a^8 = 1$ or $\mathbf{a = \pm\frac{1}{\sqrt{2}}}$. In these cases Newton's method cycles. Question: Is this where the white basin ends along the x axis?

36 By Problem 34 Newton's method diverges if $16a^8 > 1$: for instance $(x_0, y_0) = (1, 0)$ as in Example 9 in the text.

38 A famous fractal shows the three basins of attraction – see almost any book displaying fractals. Remarkable property of the boundary points between basins: **they touch all three basins!** Try to draw 3 regions with this property.

40 Problem 39 has $2x\Delta x - \Delta y = y - x^2$ and $\Delta x - \Delta y = y - x$. Subtraction gives $(2x - 1)\Delta x = x - x^2$. Then $x + \Delta x = x + \frac{x - x^2}{2x - 1} = \frac{x^2}{2x - 1}$. By the second equation this is also $y + \Delta y$. Now find the basin: If $x < 0$ then $\Delta x > 0$ but $x + \Delta x$ still < 0 : moving toward 0. If $0 \le x < \frac{1}{2}$ then $x + \Delta x < 0$. So the basin for (0,0) has **all $\mathbf{x < \frac{1}{2}}$**. The line $x = \frac{1}{2}$ gives **blowup**. If $\frac{1}{2} < x \le 1$ then $\Delta x > 0$. If $x > 1$ then $\Delta x < 0$ but $x + \Delta x = \frac{x^2}{2x - 1} \ge 1$ (because $x^2 - 2x + 1 \ge 0$). So the basin for (1,1) has **all $\mathbf{x > \frac{1}{2}}$**.

42 $J = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ is singular; g and h have the same tangent planes. Newton's equations $2\Delta x + 2\Delta y = -2$ and $\Delta x + \Delta y = -1$ have infinitely many solutions.

13.4 Directional Derivatives and Gradients (page 495)

$D_{\mathbf{u}} f$ gives the rate of change of $\mathbf{f(x,y)}$ in the direction $\mathbf{u}$. It can be computed from the two derivatives $\partial \mathbf{f}/\partial \mathbf{x}$ and $\partial \mathbf{f}/\partial \mathbf{y}$ in the special directions **(1,0)** and **(0,1)**. In terms of u_1, u_2 the formula is $D_{\mathbf{u}} f = \mathbf{f_x u_1 + f_y u_2}$. This is a dot product of $\mathbf{u}$ with the vector $(\mathbf{f_x}, \mathbf{f_y})$, which is called the **gradient**. For the linear function $f = ax + by$, the gradient is grad $f = (\mathbf{a}, \mathbf{b})$ and the directional derivative is $D_{\mathbf{u}} f = (\mathbf{a}, \mathbf{b}) \cdot \mathbf{u}$.

The gradient $\nabla f = (f_x, f_y)$ is not a vector in **three** dimensions, it is a vector in the **base plane**. It is perpendicular to the **level** lines. It points in the direction of **steepest** climb. Its magnitude $|\text{grad} f|$ is **the steepness** $\sqrt{\mathbf{f_x^2 + f_y^2}}$. For $f = x^2 + y^2$ the gradient points **out from the origin** and the slope in that steepest direction is $|(\mathbf{2x}, \mathbf{2y})| = \mathbf{2r}$.

The gradient of $f(x,y,z)$ is $(\mathbf{f_x}, \mathbf{f_y}, \mathbf{f_z})$. This is different from the gradient on the surface $F(x,y,z) = 0$, which is $-(F_x/F_z)\mathbf{i} - (F_y/F_z)\mathbf{j}$. Traveling with velocity $\mathbf{v}$ on a curved path, the rate of change of f is $df/dt = (\textbf{grad f}) \cdot \mathbf{v}$. When the tangent direction is $\mathbf{T}$, the slope of f is $df/ds = (\textbf{grad f}) \cdot \mathbf{T}$. In a straight direction $\mathbf{u}$, df/ds is the same as **the directional derivative** $D_{\mathbf{u}} f$.

1 grad $f = 2x\mathbf{i} - 2y\mathbf{j}$, $D_{\mathbf{u}} f = \sqrt{3}x - y$, $D_{\mathbf{u}} f(P) = \sqrt{3}$
3 grad $f = e^x \cos y\, \mathbf{i} - e^x \sin y\, \mathbf{j}$, $D_{\mathbf{u}} f = -e^x \sin y$, $D_{\mathbf{u}} f(P) = -1$
5 $f = \sqrt{x^2 + (y-3)^2}$, grad $f = \frac{x}{f}\mathbf{i} + \frac{y-3}{f}\mathbf{j}$, $D_{\mathbf{u}} f = \frac{x}{f}$, $D_{\mathbf{u}} f(P) = \frac{1}{\sqrt{5}}$ **7** grad $f = \frac{2x}{x^2+y^2}\mathbf{i} + \frac{2y}{x^2+y^2}\mathbf{j}$
9 grad $f = 6x\mathbf{i} + 4y\mathbf{j} = 6\mathbf{i} + 8\mathbf{j}$ = steepest direction at P; level direction $-8\mathbf{i} + 6\mathbf{j}$ is perpendicular; 10, 0
11 T; F (grad f is a vector); F;T **13** $\mathbf{u} = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})$, $D_{\mathbf{u}} f = \sqrt{a^2+b^2}$
15 grad $f = (e^{x-y}, -e^{x-y}) = (e^{-1}, -e^{-1})$ at P; $\mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$, $D_{\mathbf{u}} f = \sqrt{2}e^{-1}$
17 grad $f = \mathbf{O}$ at maximum; level curve is one point **19** $\mathbf{N} = (-1,1,-1)$, $\mathbf{U} = (-1,1,2)$, $\mathbf{L} = (1,1,0)$
21 Direction $-\mathbf{U} = (-2,0,-4)$ **23** $-\mathbf{U} = (\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, \frac{-x^2-y^2}{1-x^2-y^2})$
25 $f = (x+2y)$ and $(x+2y)^2$; $\mathbf{i} + 2\mathbf{j}$; straight lines $x + 2y$ = constant (perpendicular to $\mathbf{i} + 2\mathbf{j}$)
27 grad $f = \pm(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}})$; grad $g = \pm(2\sqrt{5}, \sqrt{5})$, $f = \pm(\frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}}) + C$, $g = \pm(2\sqrt{5}x + \sqrt{5}y) + C$
29 θ = constant along ray in direction $\mathbf{u} = \frac{3\mathbf{i}+4\mathbf{j}}{5}$; grad $\theta = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2} = \frac{-4\mathbf{i}+3\mathbf{j}}{25}$; $\mathbf{u}\cdot$ grad $\theta = 0$
31 $\mathbf{U} = (f_x, f_y, f_x^2 + f_y^2) = (-1,-2,5)$; $-\mathbf{U} = (-1,-2,5)$; tangent at the point (2,1,6)
33 grad f toward $2\mathbf{i} + \mathbf{j}$ at P, $\mathbf{j}$ at Q, $-2\mathbf{i} + \mathbf{j}$ at R; $(2, \frac{1}{2})$ and $(2\frac{1}{2}, 2)$; largest upper left, smallest lower right; $z_{\max} > 9$; z goes from 2 to 8 and back to 6
35 $f = \frac{1}{2}\sqrt{(x-1)^2 + (y-1)^2}$; $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})_{0,0} = (\frac{-1}{2\sqrt{2}}, \frac{-1}{2\sqrt{2}})$
37 Figure C now shows level curves; $|\text{grad} f|$ is varying; f could be xy
39 $x^2 + xy$; e^{x-y}; no function has $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = -x$ because then $f_{xy} \neq f_{yx}$
41 $\mathbf{v} = (1, 2t)$; $\mathbf{T} = \mathbf{v}/\sqrt{1+4t^2}$; $\frac{df}{dt} = \mathbf{v} \cdot (2t, 2t^2) = 2t + 4t^3$; $\frac{df}{ds} = (2t + 4t^3)/\sqrt{1+4t^2}$

43 $\mathbf{v} = (2,3); \mathbf{T} = \frac{\mathbf{v}}{\sqrt{13}}; \frac{df}{dt} = \mathbf{v}\cdot(2x_0 + 4t, -2y_0 - 6t) = 4x_0 - 6y_0 - 10t; \frac{df}{ds} = \frac{df/dt}{\sqrt{13}}$

45 $\mathbf{v} = (e^t, 2e^{2t}, -e^{-t}); \mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$; grad $f = (\frac{1}{x}, \frac{1}{y}, \frac{1}{z}) = (e^{-t}, e^{-2t}, e^t), \frac{df}{dt} = 1 + 2 - 1, \frac{df}{ds} = \frac{2}{|\mathbf{v}|}$

47 $\mathbf{v} = (-2\sin 2t, 2\cos 2t), \mathbf{T} = (-\sin 2t, \cos 2t)$; grad $f = (y, x), \frac{df}{ds} = -2\sin^2 2t + 2\cos^2 2t, \frac{df}{dt} = \frac{1}{2}\frac{df}{ds}$; zero slope because $f = 1$ on this path

49 $z - 1 = 2(x-4) + 3(y-5); f = 1 + 2(x-4) + 3(y-5)$ **51** grad $f \cdot \mathbf{T} = 0; \mathbf{T}$

2 grad $f = f_x\mathbf{i} + f_y\mathbf{j} = 3\mathbf{i} + 4\mathbf{j}; D_{\mathbf{u}}f = 3(\frac{3}{5}) + 4(\frac{4}{5}) = 5$ at every point P.

4 grad $f = 10y^9\mathbf{j} : D_{\mathbf{u}}f = -10y^9; D_{\mathbf{u}}f(P) = 10.$ **6** grad $f = \frac{-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$

8 grad $f = -\frac{(x-1)\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{((x-1)^2+y^2+z^2)^{3/2}} - \frac{(x+1)\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{((x+1)^2+y^2+z^2)^{3/2}}$

10 $\frac{d}{du_1}(3u_1 + \sqrt{1-u_1^2}) = 3 - \frac{u_1}{\sqrt{1-u_1^2}} = 0$ if $u_1 = 3\sqrt{1-u_1^2}$. Then $u_1^2 = 9(1-u_1^2)$ or $10u_1^2 = 9$ or $\mathbf{u_1} = \frac{3}{\sqrt{10}}$, which makes the slope equal to $\frac{3\cdot 3}{\sqrt{10}} + \sqrt{\frac{1}{10}} = \sqrt{10}$.

12 In one dimension the gradient of $f(x)$ is $\frac{df}{dx}\mathbf{i}$. The two possible directions are $\mathbf{u} = \mathbf{i}$ and $\mathbf{u} = -\mathbf{i}$. The two directional derivatives are $+\frac{\mathbf{df}}{\mathbf{dx}}$ and $-\frac{\mathbf{df}}{\mathbf{dx}}$. The normal vector $\mathbf{N}$ is $\frac{df}{dx}\mathbf{i} - \mathbf{j}$.

14 Here $f = 2x$ above the line $y = 2x$ and $f = y$ below that line. The two pieces agree on the line. Then grad $f = 2\mathbf{i}$ above and grad $f = \mathbf{j}$ below. Surprisingly f increases fastest *along* the line, which is the direction $\mathbf{u} = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$ and gives $D_{\mathbf{u}}f = \frac{2}{\sqrt{5}}$.

16 grad $f = \frac{-x\mathbf{i} - y\mathbf{j}}{\sqrt{5-x^2-y^2}} = \frac{-\mathbf{i}-2\mathbf{j}}{0}$ and P is a *rough point!* The rate of increase is **infinite** (provided $x^2 + y^2$ stays below 5; the direction must point *into this circle*).

18 (a) $\mathbf{N}\cdot\mathbf{U} = \mathbf{N}\cdot\mathbf{L} = \mathbf{U}\cdot\mathbf{L} = 0$ (b) $\mathbf{N}$ is **perpendicular** to the tangent plane, $\mathbf{U}$ and $\mathbf{L}$ are **parallel** to the tangent plane. (c) The gradient is the xy projection of $\mathbf{N}$ and also of $\mathbf{U}$. The projection of $\mathbf{L}$ points along the **level curve**.

20 $\mathbf{N} = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1), \mathbf{U} = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 1)$, and $L = (\frac{y}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}}, 0)$. $\mathbf{U}$ goes up the side of the cone.

22 $-\mathbf{U} = (-4, 3, -25)$. The xy direction of flow is $-$ grad $z = -4\mathbf{i} + 3\mathbf{j}$.

24 $-\mathbf{U} = (\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, -1)$. The xy direction of flow is radially inward.

26 $f = \frac{y}{x} = \frac{1}{1}$ is a straight level curve $y = x$. The direction of the gradient is perpendicular to that level curve: gradient along $-\mathbf{i} + \mathbf{j}$. Check: grad $f = \frac{-y}{x^2}\mathbf{i} + \frac{1}{x}\mathbf{j} = -\mathbf{i} + \mathbf{j}$.

28 (a) **False** because $f + C$ has the same gradient as f (b) **True** because the line direction $(1,1,-1)$ is also the normal direction $\mathbf{N}$ (c) **False** because the gradient is in 2 dimensions.

30 $\theta = \tan^{-1}\frac{y}{x}$ has grad $\theta = (\frac{-y/x^2}{1+(y/x)^2}, \frac{1/x}{1+(y/x)^2}) = \frac{(-y,x)}{x^2+y^2}$. The unit vector in this direction is $\mathbf{T} = (\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}})$. Then grad $\theta \cdot T = \frac{y^2+x^2}{(x^2+y^2)^{3/2}} = \frac{1}{r}$.

32 $T = e^{-x^2-y^2}$ has $\Delta T \approx \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y = (-2x\Delta x - 2y\Delta y)e^{-x^2-y^2} = (-2\Delta x + 4\Delta y)e^{-5}$. This is largest going in toward (0,0), in the direction $\mathbf{u} = (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$.

34 The gradient is $(2ax + c)\mathbf{i} + (2by + d)\mathbf{j}$. The figure shows $c = 0$ and $d \approx \frac{1}{3}$ at the origin. Then $b \approx \frac{1}{3}$ from the gradient at (0,1). Then $a \approx -\frac{1}{4}$ from the gradient at (2,0). The function $-\frac{1}{4}x^2 + \frac{1}{3}y^2 + \frac{1}{3}y$ has hyperbolas opening upwards as level curves.

36 grad f is tangent to $xy = c$ and therefore perpendicular to $y\mathbf{i} + x\mathbf{j}$. So grad f is a multiple of $x\mathbf{i} - y\mathbf{j}$. |grad f| is larger at $\mathbf{Q}$ than P. It is *not* constant on the hyperbolas. The function could be $f = x^2 - y^2$. Its level curves are also hyperbolas, perpendicular to those in the figure.

38 $f(0,1) = B + C = 0, f(1,0) = A + C = 1$, and $f(2,1) = 2A + B + C = 2$. Solution $A = 1, B = C = 0$. So grad $f = \mathbf{i}$.

40 The function is $xy + C$ so its level curves are standard hyperbolas.

42 $\mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt}) = (-2\sin 2t, 2\cos 2t); \mathbf{T} = (-\sin 2t, \cos 2t)$; grad $f = (1,0)$ so $\frac{df}{dt} = -2\sin 2t$ and $\frac{df}{ds} = -\sin 2t$.

44 $\mathbf{v} = (2t, 0)$ and $\mathbf{T} = (1,0)$; grad $f = (y, x)$ so $\frac{df}{dt} = 2ty = 6t$ and $\frac{df}{ds} = y = 3$.

46 $\mathbf{v} = (1, 2t, 3t^2)$ and $\mathbf{T} = \mathbf{v}/\sqrt{1+4t^2+9t^4}$; grad $f = (4x, 6y, 2z) = (4t, 6t^2, 2t^3)$ so $\frac{df}{dt} = 4t + 12t^3 + 6t^5$ and $\frac{df}{ds} = \frac{4t+12t^3+6t^5}{\sqrt{1+4t^2+9t^4}}$.

48 $D^2 = (x-1)^2 + (y-2)^2$ has $2D\frac{\partial D}{\partial x} = 2(x-1)$ or $\frac{\partial D}{\partial x} = \frac{x-1}{D}$. Similarly $2D\frac{\partial D}{\partial y} = 2(y-2)$ and $\frac{\partial D}{\partial y} = \frac{y-2}{D}$. Then $|\text{grad } D| = (\frac{x-1}{D})^2 + (\frac{y-2}{D})^2 = 1$. The graph of D is a 45° cone with its vertex at (1,2).

50 The directional derivative at P is the limit as $\Delta s \to 0$ of $\frac{\Delta f}{\Delta s} = \frac{f(x_0+u_1\Delta s, y_0+u_2\Delta s) - f(x_0,y_0)}{\Delta s}$. Then $\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = D_{\mathbf{u}}f(P)$ times Δs and $D_{\mathbf{u}}f(P) = u_1\frac{\partial f}{\partial x}(P) + u_2\frac{\partial f}{\partial y}(P)$.

13.5 The Chain Rule (page 503)

The chain rule applies to a function of a **function**. The x derivative of $f(g(x,y))$ is $\partial f/\partial x = (\partial \mathbf{f}/\partial \mathbf{g})(\partial \mathbf{g}/\partial \mathbf{x})$. The y derivative is $\partial f/\partial y = (\partial \mathbf{f}/\partial \mathbf{g})(\partial \mathbf{g}/\partial \mathbf{y})$. The example $f = (x+y)^n$ has $g = \mathbf{x}+\mathbf{y}$. Because $\partial g/\partial x = \partial g/\partial y$ we know that $\partial \mathbf{f}/\partial \mathbf{x} = \partial \mathbf{f}/\partial \mathbf{y}$. This **partial** differential equation is satisfied by any function of $x+y$.

Along a path, the derivative of $f(x(t), y(t))$ is $df/dt = (\partial \mathbf{f}/\partial \mathbf{x})(\mathbf{dx}/\mathbf{dt}) + (\partial \mathbf{f}/\partial \mathbf{y})(\mathbf{dy}/\mathbf{dt})$. The derivative of $f(x(t), y(t), z(t))$ is $\mathbf{f_x x_t} + \mathbf{f_y y_t} + \mathbf{f_z z_t}$. If $f = xy$ then the chain rule gives $df/dt = \mathbf{y}\ \mathbf{dx}/\mathbf{dt} + \mathbf{x}\ \mathbf{dy}/\mathbf{dt}$. That is the same as the **product** rule! When $x = u_1 t$ and $y = u_2 t$ the path is a **straight** line. The chain rule for $f(x,y)$ gives $df/dt = \mathbf{f_x u_1} + \mathbf{f_y u_2}$. That is the **directional** derivative $D_{\mathbf{u}}f$.

The chain rule for $f(x(t,u), y(t,u))$ is $\partial f/\partial t = (\partial \mathbf{f}/\partial \mathbf{x})(\partial \mathbf{x}/\partial \mathbf{t}) + (\partial \mathbf{f}/\partial \mathbf{y})(\partial \mathbf{y}/\partial \mathbf{t})$. We don't write df/dt because **f also depends on u**. If $x = r\cos\theta$ and $y = r\sin\theta$, the variables t, u change to $\mathbf{r}$ and θ. In this case $\partial f/\partial r = (\partial \mathbf{f}/\partial \mathbf{x})\cos\theta + (\partial \mathbf{f}/\partial \mathbf{y})\sin\theta$ and $\partial f/\partial \theta = (\partial \mathbf{f}/\partial \mathbf{x})(-\mathbf{r}\sin\theta) + (\mathbf{df}/\mathbf{dy})(\mathbf{r}\cos\theta)$. That connects the derivatives in **rectangular** and **polar** coordinates. The difference between $\partial r/\partial x = x/r$ and $\partial r/\partial x = 1/\cos\theta$ is because $\mathbf{y}$ is constant in the first and θ is constant in the second.

With a relation like $xyz = 1$, the three variables are **not** independent. The derivatives $(\partial f/\partial x)_y$ and $(\partial f/\partial x)_z$ and $(\partial f/\partial x)$ mean **that y is held constant, and z is constant, and both are constant.** For $f = x^2 + y^2 + z^2$ with $xyz = 1$, we compute $(\partial f/\partial x)_z$ from the chain rule $\partial \mathbf{f}/\partial \mathbf{x} + (\partial \mathbf{f}/\partial \mathbf{y})(\partial \mathbf{y}/\partial \mathbf{x})$. In that rule $\partial z/\partial x = -\mathbf{1}/\mathbf{x^2 y}$ from the relation $xyz = 1$.

1 $f_x = f_y = \cos(x+y)$ **3** $f_y = cf_x = c\cos(x+cy)$ **5** $3g^2\frac{\partial g}{\partial x}\frac{dx}{dt} + 3g^2\frac{\partial g}{\partial y}\frac{dy}{dt}$ **7** Moves left at speed 2

9 $\frac{dx}{dt} = 1$ (wave moves at speed 1)

11 $\frac{\partial^2}{\partial x^2}f(x+iy) = f''(x+iy), \frac{\partial^2}{\partial y^2}f(x+iy) = i^2 f''(x+iy)$ so $f_{xx} + f_{yy} = 0; (x+iy)^2 = (x^2-y^2) + i(2xy)$

13 $\frac{df}{dt} = 2x(1) + 2y(2t) = 2t + 4t^3$ **15** $\frac{df}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} = -1$ **17** $\frac{df}{dt} = \frac{1}{x+y}\frac{dx}{dt} + \frac{1}{x+y}\frac{dy}{dt} = 1$

19 $V = \frac{1}{3}\pi r^2 h, \frac{dV}{dt} = \frac{2\pi rh}{3}\frac{dr}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt} = 36\pi$

21 $\frac{dD}{dt} = \frac{90}{\sqrt{90^2+90^2}}(60) + \frac{90}{\sqrt{90^2+90^2}}(45) = \frac{105}{\sqrt{2}}$ mph; $\frac{dD}{dt} = \frac{60}{\sqrt{45^2+60^2}}(60) + \frac{45}{\sqrt{45^2+60^2}}(45) \approx 74$ mph

23 $\frac{df}{dt} = u_1\frac{\partial f}{\partial x} + u_2\frac{\partial f}{\partial y} + u_3\frac{\partial f}{\partial z}$ **25** $\frac{\partial f}{\partial t} = 1$ with x and y fixed; $\frac{df}{dt} = 6$

27 $f_t = f_x t + f_y(2t); f_{tt} = f_{xt}t + f_x + 2f_{yt}t + 2f_y = (f_{xx}t + f_{yx}(2t))t + f_x + 2(f_{xy}t + f_{yy}(2t))t + 2f_y$

29 $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta, \theta$ is fixed

31 $r_{xx} = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}; \frac{\partial}{\partial x}(\frac{x}{r}) = \frac{1}{r} - xr^{-2}\frac{\partial r}{\partial x} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3}$

33 $(\frac{\partial z}{\partial x})_y = \frac{1}{\sqrt{1-(x+y)^2}}; (\cos z)(\frac{\partial z}{\partial x})_y = 1$; first answer is also $\frac{1}{\sqrt{1-\sin^2 z}} = \frac{1}{\cos z}$

35 $f_r = f_x\cos\theta + f_y\sin\theta, f_{r\theta} = -f_x\sin\theta + f_y\cos\theta + f_{xx}(-r\sin\theta\cos\theta) + f_{xy}(-r\sin^2\theta + r\cos^2\theta) + f_{yy}(r\cos\theta\sin\theta)$

37 Yes (with y constant): $\frac{\partial z}{\partial x} = ye^{xy}, \frac{\partial x}{\partial z} = \frac{1}{zy} = \frac{1}{ye^{xy}}$ **39** $f_t = f_x x_t + f_y y_t; f_{tt} = f_{xx}x_t^2 + 2f_{xy}x_t y_t + f_{yy}y_t^2$

41 $(\frac{\partial f}{\partial x})_z = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} = a - \frac{3}{5}b; (\frac{\partial f}{\partial x})_y = a; (\frac{\partial f}{\partial z})_x = \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{5}b$

43 1 **45** $f = y^2$ so $f_x = 0, f_y = 2y = 2r\sin\theta; f = r^2$ so $f_r = 2r = 2\sqrt{x^2+y^2}, f_\theta = 0$

47 $g_u = f_x x_u + f_y y_u = f_x + f_y; g_v = f_x x_v + f_y y_v = f_x - f_y; g_{uu} = f_{xx}x_u + f_{xy}y_u + f_{yx}x_u + f_{yy}y_u$
$= f_{xx} + 2f_{xy} + f_{yy}; g_{vv} = f_{xx}x_v + f_{xy}y_v - f_{yx}x_v - f_{yy}y_v = f_{xx} - 2f_{xy} + f_{yy}$. Add $g_{uu} + g_{vv}$ **49** True

2 $f_x = 10a(ax+by)^9$ and $f_y = 10b(ax+by)^9; bf_x = af_y$. **4** $f_x = \frac{1}{x+7y}$ and $f_y = \frac{7}{x+7y}; 7f_x = f_y$.

6 $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ is the **product rule** $\mathbf{y\frac{dx}{dt} + x\frac{dy}{dt}}$. In terms of u and v this is $\frac{d}{dt}(uv) = v\frac{du}{dt} + u\frac{dv}{dt}$.

8 $f_{tt} = c^2n(n-1)(x+ct)^{n-2}$ which equals c^2f_{xx}. Choose $C = -\mathbf{c}: f = (x-ct)^n$ also has $f_{tt} = c^2f_{xx}$.

10 Since $\sin(0-t)$ is decreasing (it is $-\sin t$), you go **down**. At $t = 4$, your height is $-\mathbf{\sin 4}$ and your velocity is $-\cos(-4) = -\mathbf{\cos 4}$.

12 (a) $f_r = 2re^{2i\theta}, f_{rr} = 2e^{2i\theta}, f_{\theta\theta} = r^2(2i)^2e^{2i\theta}$ and $f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2} = 0$. Take real parts *throughout* to find the same for $r^2\cos 2\theta$ (and imaginary parts for $r^2\sin 2\theta$). (b) Any function $f(re^{i\theta})$ has
$f_r = e^{i\theta}f'(re^{i\theta})$ and $f_{rr} = (e^{i\theta})^2f''(re^{i\theta})$ and $f_\theta = ire^{i\theta}f'(re^{i\theta})$ and $f_{\theta\theta} = i^2re^{i\theta}f' + (ire^{i\theta})^2f''$.
Any $f(re^{i\theta})$ or any $f(x+iy)$ will satisfy the polar or rectangular form of Laplace's equation.

14 $\frac{df}{dt} = \frac{x}{\sqrt{x^2+y^2}}(1) + \frac{y}{\sqrt{x^2+y^2}}(2t) = \frac{t+2t^3}{\sqrt{t^2+t^4}} = \frac{1+2t^2}{\sqrt{1+t^2}}$.

16 Since $\frac{x}{y} = \frac{1}{2}$ we must find $\frac{df}{dt} = 0$. The chain rule gives $\frac{1}{y}\frac{dx}{dt} - \frac{x}{y^2}\frac{dy}{dt} = \frac{1}{2e^t}(e^t) - \frac{e^t}{4e^{2t}}(2e^t) = 0$.

18 $\frac{df}{dt} = (4t^3)(1) + (0)(1) = 4t^3$.

20 The rocket's position is $x = 6t, y = t^2$. Its speed from (0,0) is $\frac{d}{dt}\sqrt{(6t)^2 + (t^2)^2} = \frac{36t+2t^3}{\sqrt{(6t)^2+(t^2)^2}}$. At $t = 0$ this speed is $\frac{36}{6} = \mathbf{6}$. The rate of change of $\theta = \tan^{-1}\frac{t^2}{6t} = \tan^{-1}\frac{t}{6}$ is $\frac{\frac{1}{6}}{1+(\frac{t}{6})^2}$. At $t = 0$ this is $\mathbf{\frac{1}{6}}$.

22 Driving south $\frac{dT}{dt} = (.05)(70) = 3.5$ degrees per hour. Southeast now gives $\frac{dT}{dt} = (.05)\frac{80}{\sqrt{2}} + (.01)\frac{80}{\sqrt{2}}$ ≈ 3.4 degrees per hour. $\frac{dT}{dt}$ is larger going south.

24 $\frac{d^2f}{dt^2} = \frac{\partial f_t}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_t}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f_t}{\partial z}\frac{\partial z}{\partial t} = (f_xu_1 + f_yu_2 + f_zu_3)_xu_1 + (f_xu_1 + f_yu_2 + f_zu_3)_yu_2 + (f_xu_1 + f_yu_2 + f_zu_3)_zu_3 =$ $\mathbf{f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2 + 2f_{xz}u_1u_3 + 2f_{yz}u_2u_3 + f_{zz}u_3^2}$. For $f = xyz$ this is $2zu_1u_2 + 2yu_1u_3 + 2xu_2u_3 = \mathbf{6tu_1u_2u_3}$. Check: $f = u_1u_2u_3t^3$ and $f_{tt} = 6u_1u_2u_3t$.

26 $\frac{\partial z}{\partial x} = 2(x+y)$ and $\frac{\partial x}{\partial z} = \frac{1}{2\sqrt{z}} = \frac{1}{2(x+y)}$. Yes: The product is 1 because y is constant.

28 $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \mathbf{3(x+y)^2(t+t^2)}$ and $\frac{d^2f}{dt^2} = \mathbf{3(x+y)^2(1+2t) + 6(x+y)(t+t^2)^2}$.

30 $f_{xx} = \mathbf{90a^2(ax+by+c)^8}$ and $(at+bt+c)^{10}$ has $f_{tt} = \mathbf{90(a+b)^2(at+bt+c)^8}$. It is false that $\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial f}{\partial t}$ (we also need the term $\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$).

32 $\frac{\partial r}{\partial x} = \frac{x}{r}$ and then $\frac{\partial^2 r}{\partial y\,\partial x} = -\frac{x}{r^2}\frac{\partial r}{\partial y} = -\frac{x}{r^2}\frac{y}{r} = -\frac{xy}{r^3}$.

34 $f_x = \frac{x}{\sqrt{x^2+y^2}}; f_{xx} = \frac{\sqrt{x^2+y^2} - x^2/\sqrt{x^2+y^2}}{x^2+y^2} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{r^3}; f_{xy} = \frac{-xy}{(x^2+y^2)^{3/2}}$.

36 Yes, if y is simply held constant then the old rule continues to apply.

38 $\frac{\partial f}{\partial t}(x,y,z) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$.

40 (a) $\frac{\partial f}{\partial x} = \mathbf{2x}$ (b) $f = x^2 + y^2 + (x^2+y^2)^2$ so $\frac{\partial f}{\partial x} = \mathbf{2x + 4x(x^2+y^2)}$
(c) $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 2x + 2z(2x) = \mathbf{2x + 4x(x^2+y^2)}$ (d) y is constant for $(\frac{\partial f}{\partial x})_y$.

42 $(\frac{\partial P}{\partial V})_T = -\frac{\partial F}{\partial V}/\frac{\partial F}{\partial P}$ and similarly $(\frac{\partial V}{\partial T})_P = -\frac{\partial F}{\partial T}/\frac{\partial F}{\partial V}$ and $(\frac{\partial T}{\partial P})_V = -\frac{\partial F}{\partial P}/\frac{\partial F}{\partial T}$. Multiply these three equations: the right hand sides produce $\mathbf{-1}$.

44 $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(u)$ and $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(t)$. For $f = x^2 - 2y$ these become $\frac{\partial f}{\partial t} = 2x(1) - 2(u) = 2(t+u) - 2u = \mathbf{2t}$ and similarly $\frac{\partial f}{\partial u} = \mathbf{2u}$. Check: $f = (t+u)^2 - 2tu = t^2 + u^2$ has $f_t = 2t$ and $f_u = 2u$.

46 $\sin x + \sin y = 0$ gives $\cos x + \cos y\frac{dy}{dx} = 0$ and $-\sin x - \sin y(\frac{dy}{dx})^2 + \cos y\frac{d^2y}{dx^2} = 0$. Then $\frac{dy}{dx} = -\frac{\cos x}{\cos y}$ and $\frac{d^2y}{dx^2} = \frac{\sin x + \sin y \frac{\cos^2 x}{\cos^2 y}}{\cos y}$.

48 The c derivative of $f(cx, cy) = cf(x,y)$ is $x\frac{\partial f}{\partial x}(cx,cy) + y\frac{\partial f}{\partial y}(cx,cy) = f(x,y)$. At $c = 1$ this becomes $\mathbf{x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = f(x,y)}$. Test on $f = \sqrt{x^2+y^2}$: $x\frac{x}{\sqrt{x^2+y^2}} + y\frac{y}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$.
Test on $f = \sqrt{xy}$: $x(\frac{1}{2}\frac{y}{\sqrt{x}}) + y(\frac{1}{2}\frac{x}{\sqrt{y}}) = \sqrt{xy}$. Other examples: $f(x,y) = \sqrt{ax^2 + bxy + cy^2}$ or $f = Ax + By$ or $f = \mathbf{x^{1/4}y^{3/4}}$.

13.6 Maxima, Minima, and Saddle Points (page 512)

A minimum occurs at a **stationary** point (where $f_x = f_y = 0$) or a **rough** point (no derivative) or a **boundary** point. Since $f = x^2 - xy + 2y$ has $f_x = \mathbf{2x - y}$ and $f_y = \mathbf{2 - x}$, the stationary point is $x = \mathbf{2}, y = \mathbf{4}$. This is not a minimum, because f decreases when $y = 2x$ **increases**.

The minimum of $d^2 = (x - x_1)^2 + (y - y_1)^2$ occurs at the rough point $\mathbf{(x_1, y_1)}$. The graph of d is a **cone** and grad d is a **unit** vector that points **out from** $\mathbf{(x_1, y_1)}$. The graph of $f = |xy|$ touches bottom along the lines $\mathbf{x = 0}$ **and** $\mathbf{y = 0}$. Those are "rough lines" because the derivative **does not exist**. The maximum of d and f must occur on the **boundary** of the allowed region because it doesn't occur **inside**.

When the boundary curve is $x = x(t)$, $y = y(t)$, the derivative of $f(x,y)$ along the boundary is $\mathbf{f_x x_t + f_y y_t}$ (chain rule). If $f = x^2 + 2y^2$ and the boundary is $x = \cos t, y = \sin t$, then $df/dt = \mathbf{2 \sin t \cos t}$. It is zero at the points $\mathbf{t = 0, \pi/2, \pi, 3\pi/2}$. The maximum is at $\mathbf{(0, \pm 1)}$ and the minimum is at $\mathbf{(\pm 1, 0)}$. Inside the circle f has an absolute minimum at **(0,0)**.

To separate maximum from minimum from **saddle point**, compute the **second** derivatives at a **stationary** point. The tests for a minimum are $\mathbf{f_{xx} > 0}$ **and** $\mathbf{f_{xx}f_{yy} > f_{xy}^2}$. The tests for a maximum are $\mathbf{f_{xx} < 0}$ **and** $\mathbf{f_{xx}f_{yy} > f_{xy}^2}$. In case $ac < \mathbf{b^2}$ or $f_{xx}f_{yy} < \mathbf{f_{xy}^2}$, we have a **saddle point**. At all points these tests decide between concave up and **concave down** and "indefinite". For $f = 8x^2 - 6xy + y^2$, the origin is a **saddle point**. The signs of f at (1,0) and (1,3) are **+ and** $-$.

The Taylor series for $f(x,y)$ begins with the terms $\mathbf{f(0,0) + xf_x + yf_y + \frac{1}{2}x^2 f_{xx} + xyf_{xy} + \frac{1}{2}y^2 f_{yy}}$. The coefficient of $x^n y^m$ is $\mathbf{\partial^{n+m} f/\partial x^n \partial y^m (0,0)}$ **divided by n!m!** To find a stationary point numerically, use New-

ton's method or steepest descent.

1 (0,0) is a minimum **3** (3,0) is a saddle point **5** No stationary points **7** (0,0) is a maximum
9 (0,0,2) is a minimum **11** All points on the line $x = y$ are minima **13** (0,0) is a saddle point
15 (0,0) is a saddle point; (2,0) is a minimum; $(0,-2)$ is a maximum; $(2,-2)$ is a saddle point
17 Maximum of area $(12-3y)y$ is 12
19 $\begin{matrix} 2(x+y)+2(x+2y-5)+2(x+3y-4)=0 \\ 2(x+y)+4(x+2y-5)+6(x+3y-4)=0 \end{matrix}$ gives $\begin{matrix} x=2; \\ y=-1 \end{matrix}$ min because $E_{xx}E_{yy} = (6)(28) > E_{xy}^2 = 12^2$
21 Minimum at $(0, \frac{1}{2})$; $(0,1)$; $(0,1)$
23 $\frac{df}{dt} = 0$ when $\tan t = \sqrt{3}$; $f_{\max} = 2$ at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $f_{\min} = -2$ at $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$
25 $(ax+by)_{\max} = \sqrt{a^2+b^2}$; $(x^2+y^2)_{\min} = \frac{1}{a^2+b^2}$ **27** $0 < c < \frac{1}{4}$
29 The vectors head-to-tail form a 60-60-60 triangle. The outer angle is 120° **31** $2+\sqrt{3}; 1+\sqrt{3}; 1+\frac{\sqrt{3}}{10}$
35 Steiner point where the arcs meet **39** Best point for $p = \infty$ is equidistant from corners
41 grad $f = (\sqrt{2}\,\frac{x-x_1}{d_1} + \frac{x-x_2}{d_2} + \frac{x-x_3}{d_3}, \sqrt{2}\,\frac{y-y_1}{d_1} + \frac{y-y_2}{d_2} + \frac{y-y_3}{d_3})$; angles are 90-135-135
43 Third derivatives all 6; $f = \frac{6}{3!}x^3 + \frac{6}{2!}x^2y + \frac{6}{2!}xy^2 + \frac{6}{3!}y^3$
45 $(\frac{\partial}{\partial x})^n(\frac{\partial}{\partial y})^m \ln(1-xy)]_{0,0} = n!(n-1)!$ for $m = n > 0$, other derivatives zero; $f = -xy - \frac{x^2y^2}{2} - \frac{x^3y^3}{3} - \cdots$
47 All derivatives are e^2 at (1,1); $f \approx e^2[1 + (x-1) + (y-1) + \frac{1}{2}(x-1)^2 + (x-1)(y-1) + \frac{1}{2}(y-1)^2]$
49 $x = 1, y = -1 : f_x = 2, f_y = -2, f_{xx} = 2, f_{xy} = 0, f_{yy} = 2$; series must recover $x^2 + y^2$
51 Line $x - 2y =$ constant; $x + y =$ constant
53 $\frac{x^2}{2}f_{xx} + xyf_{xy} + \frac{y^2}{2}f_{yy}]_{0,0}$; $f_{xx} > 0$ and $f_{xx}f_{yy} > f_{xy}^2$ at (0,0); $f_x = f_y = 0$ **55** $\Delta x = -1, \Delta y = -1$
57 $f = x^2(12-4x)$ has $f_{\max} = 16$ at (2,4); line has slope -4, $y = \frac{16}{x^2}$ has slope $\frac{-32}{8} = -4$
59 If the fence were not perpendicular, a point to the left or right would be closer

2 $f_x = y-1, f_y = x-1; b^2 - ac = 1; (\mathbf{1},\mathbf{1})$ is a *saddle point*
4 $f_x = 2x, f_y = -2y+4; b^2 - ac = 1; (\mathbf{0},\mathbf{2})$ is a *saddle point*
6 $f_x = e^y - e^x, f_y = xe^y; (\mathbf{0},\mathbf{0})$ is the stationary point; $f_{xx} = -e^x = -1, f_{xy} = e^y = 1, f_{yy} = xe^y = 0$ so $b^2 - ac = 1$: *saddle point*
8 $f_x = 2(x+y) + 2(x+2y-6), f_y = 2(x+y) + 4(x+2y-6); (-\mathbf{6},\mathbf{6})$ is the stationary point: $f_{xx} = 4$, $f_{xy} = 6, f_{yy} = 10$ give $b^2 - ac = -4$: *minimum*
10 $f_x = x+2y-6+x+y$ and $f_y = x+2y-6+2(x+y); (-\mathbf{6},\mathbf{6})$ is the stationary point; $f_{xx} = 2, f_{xy} = 3, f_{yy} = 4$ give $b^2 - ac = 9 - 8 = 1$: *saddle point*
12 $f_x = \frac{2x}{1+y^2}$ and $f_y = \frac{-2y(1+x^2)}{(1+y^2)^2}; (\mathbf{0},\mathbf{0})$ is the stationary point; $f_{xx} = \frac{2}{1+y^2} = 2, f_{xy} = \frac{-4xy}{(1+y^2)^2} = 0$, $f_{yy} = \frac{-2(1+x^2)}{(1+y^2)^2} + \frac{8y^2(1+x^2)}{(1+y^2)^3} = -2; b^2 - ac = 4$: *saddle point*
14 $f_x = \cos x$ and $f_y = \sin y$; stationary points have $\mathbf{x} = \frac{\pi}{2} + \mathbf{n}\pi$ and $\mathbf{y} = \mathbf{m}\pi$; **maximum** when $f = 2$, *saddle point* when $f = 0$, *minimum* when $f = -2$
16 $f_x = 8y - 4x^3$ and $f_y = 8x - 4y^3$; stationary points are $(\mathbf{0},\mathbf{0})$ = *saddle point*, $(\sqrt{\mathbf{2}}, \sqrt{\mathbf{2}})$ = *maximum*, $(-\sqrt{\mathbf{2}}, -\sqrt{\mathbf{2}})$ = *minimum*.
18 Volume $= xyz = xy(1-3x-2y) = xy - 3x^2 - 2xy^2; V_x = y - 6x - 2y^2$ and $V_y = x - 4xy$; at $(0, \frac{1}{2}, 0)$ and $(\frac{1}{3}, 0, 0)$ and $(0,0,1)$ the volume is $V = 0$ (minimum); at $(\frac{1}{48}, \frac{12}{48}, \frac{21}{48})$ the volume is $V = \frac{\mathbf{7}}{\mathbf{3072}}$ (*maximum*)
20 Minimize $f(x,y) = (x-y-1)^2 + (2x+y+1)^2 + (x+2y-1)^2 : \frac{\partial f}{\partial x} = 2(x-y-1) + 4(2x+y+1) + 2(x+2y-1) = 0$ and $\frac{\partial f}{\partial y} = -2(x-y-1) + 2(2x+y+1) + 4(x+2y-1) = 0$. Solution: $\mathbf{x} = \mathbf{y} = \mathbf{0}$!
22 $\frac{\partial f}{\partial x} = 2x + 2$ and $\frac{\partial f}{\partial y} = 2y + 4$. (a) Stationary point $(-1,-2)$ yields $f_{\min} = -\mathbf{5}$. (b) On the boundary $y = 0$ the minimum of $x^2 + 2x$ is $-\mathbf{1}$ at $(-1, 0)$ (c) On the boundary $x \ge 0, y \ge 0$ the minimum is $\mathbf{0}$ at (0,0).

24 $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{5}{2} - \sqrt{2}; f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{5}{2} + \sqrt{2} = f_{\max}; f(1,0) = f(0,1) = 1 = f_{\min}$

26 $f_x = x^3 - y = 0$ and $f_y = y^3 - x = 0$ combine into $y = x^3 = y^9$. Then $y = y^9$ gives $y = 1$ or -1 or 0. At those points $f_{\min} = -\frac{1}{2}$ and $f = 0$ (relative maximum). These equations $x^3 = y, y^3 = x$ are solved by Newton's method in Section 13.3 (the basins are on the front cover).

28 $d_1 = x, d_2 = d_3 = \sqrt{(1-x)^2+1}, \frac{d}{dx}(x + 2\sqrt{(1-x)^2+1}) = 1 + \frac{2(x-1)}{\sqrt{(1-x)^2+1}} = 0$ when $(1-x)^2 + 1 = 4(x-1)^2$ or $1 - x = \frac{1}{\sqrt{3}}$ or $x = 1 - \frac{1}{\sqrt{3}}$. From that point to (1,1) the line goes up 1 and across $\frac{1}{\sqrt{3}}$, a 60° angle with the horizontal that confirms three 120° angles.

30 $d_1 = \sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}$ and then grad $d_1 = (\frac{x-x_1}{d_1}, \frac{y-y_1}{d_1}, \frac{z-z_1}{d_1})$ has length $|\text{grad } d_1| = 1$. This gradient of d_1 points directly away from (x_1, y_1, z_1). The gradient of $f = d_1 + d_2 + d_3 + d_4$ is a sum of 4 unit vectors. The sum is zero for $\frac{1}{\sqrt{3}}(1,-1,-1), \frac{1}{\sqrt{3}}(-1,-1,1), \frac{1}{\sqrt{3}}(-1,1,-1), \frac{1}{\sqrt{3}}(1,1,1)$. The equal angles have $\cos\theta = -\frac{1}{3}$ by Problem 45 of Section 11.1.

32 From an outside point the lines to the three vertices give two angles that add to less than 180°. So they cannot both be 120° as a Steiner point requires.

34 From the point $C = (0, -\sqrt{3})$ the lines to $(-1,0)$ and $(1,0)$ make a 60° angle. C is the **center** of the circle $x^2 + (y - \sqrt{3})^2 = 4$ through those two points. From any point on that circle, the lines to $(-1,0)$ and (1,0) make an angle of $2 \times 60° = 120°$. Theorem from geometry: angle from circle $= 2 \times$ angle from center.

40 The vertices are (0,0), (1,0), and (0,1). The point $(\frac{1}{2}, \frac{1}{2})$ is an equal distance $(\frac{1}{\sqrt{2}})$ from all three vertices. Note: In any triangle the intersection of the altitudes (perpendicular to edges at their midpoints) is equally distant from the vertices. If it is in the triangle, it is the best point with $p = \infty$: it minimizes the largest distance.

42 For two points, $d_1 + d_2$ is a minimum at *all points on the line between them*. (Note equal 180° angles from the vertices!) For three points, the **corner with largest angle** is the best corner.

44 $\frac{\partial^{n+m}}{\partial x^n \partial y^m}(xe^y) = xe^y$ for $n = 0, e^y$ for $n = 1$, zero for $n > 1$. Taylor series $xe^y = \mathbf{x} + \mathbf{xy} + \frac{1}{2!}\mathbf{xy}^2 + \frac{1}{3!}\mathbf{xy}^3 + \cdots$

46 All derivatives equal 1 at (0,0). Quadratic $= \mathbf{1} + \mathbf{x} + \mathbf{y} + \frac{1}{2}\mathbf{x}^2 + \mathbf{xy} + \frac{1}{2}\mathbf{y}^2$.

48 $\frac{\partial}{\partial x}(\sin x \cos y) = 1$ at (0,0) but $f = f_y = f_{xx} = f_{xy} = f_{yy} = 0$. Quadratic $= \mathbf{x}$. Check: $\sin x \cos y \approx (x - \frac{x^3}{6} + \cdots)(1 - \frac{y^2}{2} + \cdots) = x$ to quadratic accuracy.

50 $f(x+h, y+k) \approx f(x,y) + h\frac{\partial f}{\partial x}(x,y) + k\frac{\partial f}{\partial y}(x,y) + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}(x,y) + hk\frac{\partial^2 f}{\partial x\,\partial y}(x,y) + \frac{k^2}{2}\frac{\partial^2 f}{\partial y^2}(x,y)$

52 $(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z})(0,0,0)$; then $(\frac{x^2}{2}f_{xx} + \frac{y^2}{2}f_{yy} + \frac{z^2}{2}f_{zz} + xyf_{xy} + xzf_{xz} + yzf_{yz})(0,0,0)$

54 $f = (1-2s)^2 + 2(1-4s)^2$ has $\frac{df}{ds} = 0$ at $s = \frac{3}{10}$. Step ends at $x = 1 - 2s = \frac{4}{10}, y = 1 - 4s = -\frac{2}{10}$.

56 A maximum has $f_{xx} < 0$ and $f_{yy} < 0$, so they cannot add to zero. A minimum has $f_{xx} > 0$ and $f_{yy} > 0$. The functions xy and $x^2 - y^2$ solve $f_{xx} + f_{yy} = 0$ and have saddle points.

58 A house costs p, a yacht costs q : $\frac{d}{dx}f(x, \frac{k-px}{q}) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}(-\frac{p}{q}) = 0$ gives $-\frac{\partial f}{\partial x}/\frac{\partial f}{\partial y} = -\frac{p}{q}$.

13.7 Constraints and Lagrange Multipliers (page 519)

A restriction $g(x,y) = k$ is called a **constraint**. The minimizing equations for $f(x,y)$ subject to $g = k$ are $\partial \mathbf{f}/\partial \mathbf{x} = \lambda\partial \mathbf{g}/\partial \mathbf{x}, \partial \mathbf{f}/\partial \mathbf{y} = \lambda\partial \mathbf{g}/\partial \mathbf{y}$, and $\mathbf{g} = \mathbf{k}$. The number λ is the Lagrange **multiplier**. Geometrically, grad f is **parallel** to grad g at the minimum. That is because the **level** curve $f = f_{\min}$ is **tangent** to the constraint curve $g = k$. The number λ turns out to be the derivative of $f_{\min}$ with respect to $\mathbf{k}$. The Lagrange function is

$L = \mathbf{f(x,y)} - \lambda(\mathbf{g(x,y)} - \mathbf{k})$ and the three equations for x, y, λ are $\partial\mathbf{L}/\partial\mathbf{x} = \mathbf{0}$ and $\partial\mathbf{L}/\partial\mathbf{y} = \mathbf{0}$ and $\partial\mathbf{L}/\partial\lambda = \mathbf{0}$.

To minimize $f = x^2 - y$ subject to $g = x - y = 0$, the three equations for x, y, λ are $\mathbf{2x} = \lambda, -\mathbf{1} = -\lambda$, $\mathbf{x} - \mathbf{y} = \mathbf{0}$. The solution is $\mathbf{x} = \frac{1}{2}, \mathbf{y} = \frac{1}{2}, \lambda = \mathbf{1}$. In this example the curve $f(x,y) = f_{\min} = -\frac{1}{4}$ is a **parabola** which is **tangent** to the line $g = 0$ at $(x_{\min}, y_{\min})$.

With two constraints $g(x,y,z) = k_1$ and $h(x,y,z) = k_2$ there are **two** multipliers λ_1 **and** λ_2. The five unknowns are $\mathbf{x}, \mathbf{y}, \mathbf{z}, \lambda_1$, **and** λ_2. The five equations are $\mathbf{f_x} = \lambda_1\mathbf{g_x} + \lambda_2\mathbf{h_x}, \mathbf{f_y} = \lambda_1\mathbf{g_y} + \lambda_2\mathbf{h_x}$, $\mathbf{f_z} = \lambda_1\mathbf{g_z} + \lambda_2\mathbf{h_z}, \mathbf{g} = \mathbf{0}$, **and** $\mathbf{h} = \mathbf{0}$. The level surface $f = f_{\min}$ is **tangent** to the curve where $g = k_1$ and $h = k_2$. Then grad f is **perpendicular** to this curve, and so are grad g and **grad h**. With nine variables and six constraints, there will be **six** multipliers and eventually **15** equations. If a constraint is an **inequality** $g \le k$, then its multiplier must satisfy $\lambda \le 0$ at a minimum.

1 $f = x^2 + (k-2x)^2$; $\frac{df}{dx} = 2x - 4(k-2x) = 0$; $(\frac{2k}{5}, \frac{k}{5})$, $\frac{k^2}{5}$ **3** $\lambda = -4, x_{\min} = 2, y_{\min} = 2$

5 $\lambda = \frac{1}{3(4)^{1/3}} : (x,y) = (\pm 2^{1/6}, 0)$ or $(0, \pm 2^{1/6})$, $f_{\min} = 2^{1/3}$; $\lambda = \frac{1}{3} : (x,y) = (\pm 1, \pm 1)$, $f_{\max} = 2$

7 $\lambda = \frac{1}{2}, (x,y) = (2,-3)$; tangent line is $2x - 3y = 13$

9 $(1-c)^2 + (-a-c)^2 + (2-a-b-c)^2 + (2-b-c)^2$ is minimized at $a = -\frac{1}{2}, b = \frac{3}{2}, c = \frac{3}{4}$

11 $(1,-1)$ and $(-1,1)$; $\lambda = -\frac{1}{2}$

13 f is not a minimum when C crosses to lower level curve; stationary point when C is tangent to level curve

15 Substituting $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$ and $L = f_{\min}$ leaves $\frac{df_{\min}}{dk} = \lambda$

17 x^2 is never negative; $(0,0)$; $1 = \lambda(-3y^2)$ but $y = 0$; $g = 0$ has a cusp at $(0,0)$

19 $2x = \lambda_1 + \lambda_2, 4y = \lambda_1, 2z = \lambda_1 - \lambda_2, x + y + z = 0, x - z = 1$ gives $\lambda_1 = 0, \lambda_2 = 1, f_{\min} = \frac{1}{2}$ at $(\frac{1}{2}, 0, -\frac{1}{2})$

21 $(1,0,0); (0,1,0); (\lambda_1, \lambda_2, 0); x = y = 0$ **23** $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$; $\lambda = 0$

25 $(1,0,0), (0,1,0), (0,0,1)$; at these points $f = 4$ and -2 (min) and 5(max)

27 By increasing k, more points are available so $f_{\max}$ goes up. Then $\lambda = \frac{df_{\min}}{dk} \ge 0$

29 $(0,0)$; $\lambda = 0$; $f_{\min}$ stays at 0

31 $5 = \lambda_1 + \lambda_2, 6 = \lambda_1 + \lambda_3, \lambda_2 \ge 0, \lambda_3 \le 0$; subtraction $5 - 6 = \lambda_2 - \lambda_3$ or $-1 \ge 0$ (impossible); $x = 2004, y = -2000$ gives $5x + 6y = -1980$

33 $2x = 4\lambda_1 + \lambda_2, 2y = 4\lambda_1 + \lambda_3, \lambda_2 \ge 0, \lambda_3 \ge 0, 4x + 4y = 40$; max area 100 at (10,0)(0,10); min 25 at (5,5)

2 $x^2 + y^2 = 1$ and $2xy = \lambda(2x)$ and $x^2 = \lambda(2y)$ yield $2\lambda^2 + \lambda^2 = 1$. Then $\lambda = \frac{1}{\sqrt{3}}$ gives $x_{\max} = \pm\frac{\sqrt{6}}{3}$, $y_{\max} = \frac{\sqrt{3}}{3}, \mathbf{f}_{\max} = \frac{\mathbf{2\sqrt{3}}}{\mathbf{9}}$. Also $\lambda = -\frac{1}{\sqrt{3}}$ gives $\mathbf{f}_{\min} = -\frac{\mathbf{2\sqrt{3}}}{\mathbf{9}}$.

4 $x^2 + 9y^2 = 1$ and $3 = \lambda(2x)$ and $1 = \lambda(18y)$ give $\frac{1}{\lambda^2}(\frac{9}{4} + 9 \cdot \frac{1}{18^2}) = 1$ or $\lambda^2 = \frac{41}{18}$. Then $x_{\max} = \frac{9}{\sqrt{82}}$, $y_{\max} = \frac{1}{3\sqrt{82}}, \mathbf{f}_{\max} = \frac{\sqrt{\mathbf{82}}}{\mathbf{3}}$. Change signs for $(x,y,f)_{\min}$. Second approach: Fix $3x + y$ and maximize $x^2 + 9y^2$.

6 $1 = \frac{\lambda}{3}(\frac{y}{k})^{2/3}$ and $1 = \frac{2\lambda}{3}(\frac{x}{y})^{1/3}$ yield $1 = \frac{\lambda}{3}(\frac{2\lambda}{3})^2$ or $\frac{\lambda}{3} = (4)^{-1/3}$. Then $\frac{x}{y} = (\frac{3}{2\lambda})^3 = \frac{4}{8}$ so $y = 2x$. The constraint gives $x^{1/3}(2x)^{2/3} = k$ or $x = k(4)^{-1/3}$ and then $y = 2k(4)^{-1/3}$. Then $f = x + y = 3k(4)^{-1/3}$.

8 $a = \lambda(2x), b = \lambda(2y), c = \lambda(2z)$ give $\frac{1}{4\lambda^2}(a^2 + b^2 + c^2) = k^2$ and $\lambda = \sqrt{\mathbf{a^2} + \mathbf{b^2} + \mathbf{c^2}}/\mathbf{2k}$. Then $x_{\max} = ak/\sqrt{a^2+b^2+c^2}, y_{\max} = bk/\sqrt{a^2+b^2+c^2}$, and $z_{\max} = ck/\sqrt{a^2+b^2+c^2}$. Thus $(a,b,c) \cdot (x,y,z) \le \mathbf{f}_{\max} = \sqrt{\mathbf{a^2} + \mathbf{b^2} + \mathbf{c^2}}\,\mathbf{k}$ is the Schwarz inequality.

10 The base is b, the rectangle height is a, the triangle height is h, the area is $ab + \frac{1}{2}bh = 1$. Minimize $f = b + 2a + 2\sqrt{b^2/4 + h^2}$. The $\frac{\partial}{\partial a}, \frac{\partial}{\partial h}, \frac{\partial}{\partial b}$ equations are $\mathbf{2} = \lambda\mathbf{b}$, $\frac{2h}{\sqrt{b^2/4+h^2}} = \lambda(\frac{1}{2}b)$,

$1 + \frac{b/2}{\sqrt{b^2/4+h^2}} = \lambda(a + \frac{1}{2}h)$. Put $\lambda b = 2$ in the second equation and square: $(2h)^2 = \frac{b^2}{4} + h^2$ or $\mathbf{h^2 = b^2/12}$.
The third equation becomes $1 + \frac{1/2}{\sqrt{1/3}} = \lambda(a + \frac{b}{2\sqrt{12}}) = \lambda a + \frac{1}{\sqrt{12}}$. Then $\boldsymbol{\lambda a = 1 + \frac{\sqrt{3}}{3}}$.
The area is $\frac{1}{\lambda^2}[(1 + \frac{\sqrt{3}}{3})(2) + \frac{1}{2}(2)\frac{1}{\sqrt{3}}] = 1$ so $\boldsymbol{\lambda^2 = 2 + \sqrt{3}}$. This gives b, h, and a. (Not an easy problem!)

12 $y = \lambda(2x)$ and $x = \lambda(2y)$ require $2\lambda = 1$ or $2\lambda = -1$. Then $y = \pm x$. The equation $x^2 + y^2 = 2$ gives $x^2 = 1$. The maximum is at $\mathbf{x = 1, y = 1}$ or $\mathbf{x = -1, y = -1}$.

14 (a) $y - 1 = \lambda(2x), x - 1 = \lambda(2y), x^2 + y^2 = 1$ (b) $x = y = \frac{1}{1-2\lambda}$ (c) At $\lambda = -\frac{1}{2}$ both equations become $x + y = 1$ and we find the minimum points **(1,0)** and **(0,1)** where $x \neq y$.

16 Those equations come from the **chain rule**: $\frac{dg}{dx} = 0$ along the curve because $g =$ **constant**. Together the two equations give $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} - \lambda\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = \mathbf{0}$ (the Lagrange equation).

18 $f = 2x + y = 1001$ at the point $x = 1000, y = -999$. The Lagrange equations are $2 = \lambda$ and $1 = \lambda$ (no solution). Linear functions with linear constraints generally have no maximum.

20 (a) $yz = \lambda, xz = \lambda, xy = \lambda$, and $x + y + z = k$ give $x = y = z = \frac{k}{3}$ and $\lambda = \frac{k^2}{9}$ (b) $\mathbf{V_{max}} = (\frac{\mathbf{k}}{\mathbf{3}})^{\mathbf{3}}$ so $\partial V_{\max}/\partial k = \mathbf{k^2/9}$ (which is λ!) (c) Approximate $\Delta V = \lambda$ times $\Delta k = \frac{108^2}{9}(111 - 108) = \mathbf{3888}$ in^3. Exact $\Delta V = (\frac{111}{3})^3 - (\frac{108}{3})^3 = \mathbf{3677}$ in^3.

22 $2x = \lambda a, 2y = \lambda b, 2z = \lambda c$ and $ax + by + cz = d$ give $\lambda(a^2 + b^2 + c^2) = 2d$ and $x = \frac{a\lambda}{2} = \frac{ad}{a^2+b^2+c^2}$. Similarly $y = \frac{bd}{a^2+b^2+c^2}$ and $z = \frac{cd}{a^2+b^2+c^2}$. Then $\mathbf{f_{min}} = \frac{\mathbf{d^2}}{\mathbf{a^2+b^2+c^2}}$ is the *square* of the minimum distance.

24 $3 = \lambda_1 + \lambda_2$ and $5 = 2\lambda_1 + \lambda_3$ with $\lambda_2 \geq 0$ and $\lambda_3 \geq 0$. $\lambda_2 = 0$ is impossible because then $\lambda_1 = 3, \lambda_3 = -1$. So $\boldsymbol{\lambda_3 = 0}, \lambda_1 = \frac{5}{2}, \lambda_2 = \frac{1}{2}$. The minimum is $\mathbf{f = 10}$ at $\mathbf{x = 0, y = 2}$. (Note $\lambda = 0$ goes with $y \neq 0$.)

26 Reasoning: By increasing k, *more* points satisfy the constraints. *More* points are available to minimize f. Therefore $f_{\min}$ goes **down**.

28 $\lambda = 0$ when $h > k$ (not $h = k$) at the minimum. Reasoning: An increase in k leaves the same minimum. Therefore $f_{\min}$ is **unchanged**. Therefore $\lambda = df_{\min}/dk$ is **zero**.

30 $f = x^2 + y^2, x + y \geq 4$ has minimum at $\mathbf{x = y = 2}$. From $2x = \lambda(1)$ and $2y = \lambda(1)$, the multiplier is $\lambda = 4$ and $f_{\min} = 8$. Change to $x + y \geq 4 + dk$. Then $f_{\min} = 8 + \lambda dk = \mathbf{8 + 4dk}$. Check: $x_{\min} = y_{\min} = \frac{1}{2}(4 + dk)$ give $f_{\min} = (\frac{1}{2})^2(4 + dk)^2(2) = 8 + 4dk + \frac{1}{2}(dk)^2$.

32 Lagrange equations: $2 = \lambda_1 + \lambda_2, 3 = \lambda_1 + \lambda_3, 4 = \lambda_1 + \lambda_4$. Then $\lambda_4 > \lambda_3 > \lambda_2 \geq 0$. We need $\lambda_4 > 0$ and $\lambda_3 > 0$ (*correction:* not $= 0$). Zero multiplier goes with nonzero $\mathbf{x = 1}$. Nonzero multipliers go with $y = z = 0$. Then $f_{\min} = 2$. (We can see directly that $f_{\min} = 2$.)