CHAPTER 12 MOTION ALONG A CURVE

12.1 The Position Vector (page 452)

The position vector $\mathbf{R}(\mathbf{t})$ along the curve changes with the parameter t. The velocity is $\mathbf{dR/dt}$. The acceleration is $\mathbf{d^2R/dt^2}$. If the position is $\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, then $\mathbf{v} = \mathbf{j} + \mathbf{2t\,k}$ and $\mathbf{a} = \mathbf{2k}$. In that example the speed is $|\mathbf{v}| = \sqrt{\mathbf{1 + 4t^2}}$. This equals $\mathbf{ds/dt}$, where s measures **the distance along the curve.** Then $s = \int (\mathbf{ds/dt})\mathbf{dt}$. The tangent vector is in the same direction as the **velocity**, but $\mathbf{T}$ is a **unit vector.** In general $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ and in the example $\mathbf{T} = (\mathbf{j} + \mathbf{2t\,k})/\sqrt{\mathbf{1+4t^2}}$.

Steady motion along a line has $\mathbf{a} =$ **zero.** If the line is $x = y = z$, the unit tangent vector is $\mathbf{T} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{\mathbf{3}}$. If the speed is $|\mathbf{v}| = \sqrt{3}$, the velocity vector is $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. If the initial position is (1,0,0), the position vector is $\mathbf{R}(t) = (\mathbf{1 + t})\mathbf{i} + \mathbf{t\,j} + \mathbf{t\,k}$. The general equation of a line is $x = x_0 + tv_1, y = \mathbf{y_0} + \mathbf{tv_2}, z = \mathbf{z_0} + \mathbf{t\,v_3}$. In vector notation this is $\mathbf{R}(t) = \mathbf{R_0} + \mathbf{t\,v}$. Eliminating t leaves the equations $(x - x_0)/v_1 = (y - y_0)/v_2 = (\mathbf{z} - \mathbf{z_0})/\mathbf{v_3}$. A line in space needs **two** equations where a plane needs **one.** A line has one parameter where a plane has **two.** The line from $\mathbf{R_0} = (1,0,0)$ to (2,2,2) with $|\mathbf{v}| = 3$ is $\mathbf{R(t)} = (\mathbf{1+t})\mathbf{i} + \mathbf{2t\,j} + \mathbf{2t\,k}$.

Steady motion around a circle (radius r, angular velocity ω) has $x = \mathbf{r}\cos\omega\mathbf{t}, y = \mathbf{r}\sin\omega\mathbf{t}, z = 0$. The velocity is $\mathbf{v} = -\mathbf{r}\omega\sin\omega\mathbf{t}\,\mathbf{i} + \mathbf{r}\,\omega\cos\omega\mathbf{t}\,\mathbf{j}$. The speed is $|\mathbf{v}| = \mathbf{r}\omega$. The acceleration is $\mathbf{a} = -\mathbf{r}\omega^2(\cos\omega\mathbf{t}\,\mathbf{i} + \sin\omega\mathbf{t}\,\mathbf{j})$, which has magnitude $\mathbf{r}\omega^2$ and direction **toward** $(0,0)$. Combining upward motion $\mathbf{R} = t\mathbf{k}$ with this circular motion produces motion around a **helix.** Then $\mathbf{v} = -\mathbf{r}\omega\sin\omega\mathbf{t}\,\mathbf{i} + \mathbf{r}\omega\cos\omega\mathbf{t}\,\mathbf{j} + \mathbf{k}$ and $|\mathbf{v}| = \sqrt{\mathbf{1 + r^2\omega^2}}$.

1 $\mathbf{v}(1) = \mathbf{i} + 3\mathbf{j}$; speed $\sqrt{10}$; **3** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t}$; tangent to circle is perpendicular to $\frac{x}{y} = \frac{\cos t}{\sin t}$

5 $\mathbf{v} = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j} = \mathbf{i} - \mathbf{j}; y - 1 = -(x-1); xy = 1$

7 $\mathbf{R} = (1,2,4) + (4,3,0)t; \mathbf{R} = (1,2,4) + (8,6,0)t; \mathbf{R} = (5,5,4) + (8,6,0)t$

9 $\mathbf{R} = (2+t, 3, 4-t); \mathbf{R} = (2 + \frac{t^2}{2}, 3, 4 - \frac{t^2}{2})$; the same line

11 Line; $y = 2 + 2t, z = 2 + 3t; y = 2 + 4t, z = 2 + 6t$

13 Line; $\sqrt{36+9+4} = 7$; $(6,3,2)$; line segment **15** $\frac{\sqrt{2}}{2}; 1; \frac{\sqrt{2}}{2}$ **17** $x = t, y = mt + b$

19 $\mathbf{v} = \mathbf{i} - \frac{1}{t^2}\mathbf{j}, |\mathbf{v}| = \sqrt{1 + t^{-4}}, \mathbf{T} = \mathbf{v}/|\mathbf{v}|; \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}; |\mathbf{v}| = \sqrt{1+t^2}$;
$\mathbf{T} = \mathbf{v}/|\mathbf{v}|; \mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, |\mathbf{v}| = 3, \mathbf{T} = \frac{1}{3}\mathbf{v}$

21 $\mathbf{R} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$ + any $\mathbf{R}_0$; same $\mathbf{R}$ plus any $\mathbf{w}t$

23 $\mathbf{v} = (1 - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}; |\mathbf{v}| = \sqrt{2 - 2\sin t - 2\cos t}, |\mathbf{v}|_{\min} = \sqrt{2 - 2\sqrt{2}}, |\mathbf{v}|_{\max} = \sqrt{2 + 2\sqrt{2}}$;
$\mathbf{a} = -\cos t\,\mathbf{i} + \sin t\,\mathbf{j}, |\mathbf{a}| = 1$; center is on $x = t, y = t$

25 Leaves at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}); \mathbf{v} = (-\sqrt{2}, \sqrt{2}); \mathbf{R} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) + v(t - \frac{\pi}{8})$

27 $\mathbf{R} = \cos\frac{s}{\sqrt{2}}\mathbf{i} + \sin\frac{s}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$

29 $\mathbf{v} = \sec^2 t\,\mathbf{i} + \sec t\tan t\,\mathbf{j}; |\mathbf{v}| = \sec^2 t\sqrt{1 + \sin^2 t}; \mathbf{a} = 2\sec^2 t\tan t\,\mathbf{i} + (\sec^3 t + \sec t\tan^2 t)\,\mathbf{j}$;
curve is $y^2 - x^2 = 1$; hyperbola has asymptote $y = x$

31 If $\mathbf{T} = \mathbf{v}$ then $|\mathbf{v}| = 1$; line $\mathbf{R} = t\mathbf{i}$ or helix in Problem 27

33 $(x(t), y(t)) = \begin{matrix} (2t,0) & 0 \le t \le \frac{1}{2} & (3-2t,1) & 1 \le t \le \frac{3}{2} \\ (1,2t-1) & \frac{1}{2} \le t \le 1 & (0,4-2t) & \frac{3}{2} \le t \le 2 \end{matrix}$

35 $x(t) = 4\cos\frac{t}{2}, y(t) = 4\sin\frac{t}{2}$ **37** F; F; T; T; F **39** $\frac{y}{x} = \tan\theta$ but $\frac{y}{x} \ne \tan t$

41 $\mathbf{v}$ and $\mathbf{w}$; $\mathbf{v}$ and $\mathbf{w}$ and $\mathbf{u}$; $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{w}$ and $\mathbf{u}$; not zero

43 $\mathbf{u} = (8,3,2)$; projection perpendicular to $\mathbf{v} = (1,2,2)$ is $(6,-1,-2)$ which has length $\sqrt{41}$

45 $x = G(t), y = F(t); y = x^{2/3}; t = 1$ and $t = -1$ give the same x so they would give the same $y; y = G(F^{-1}(x))$

2 The path is the line $x + y = 2$. The speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{2}$.

4 $\frac{dy}{dt} = 6 - 2t = 0$ at $t = 3$, so the highest point is $\mathbf{x = 18, y = 9}$. The curve is the **parabola** $y = x - (\frac{x}{6})^2$, and $\mathbf{a = -2tj}$.

6 (a) $\mathbf{x^2 = y}$ so this is a parabola (b) $x^3 = y^2$ so $\mathbf{y = x^{3/2}}$ is a power curve (c) $\ln x = t \ln 4$ so $\mathbf{y = \frac{4}{\ln 4} x}$ is a logarithmic curve.

8 The direction of the line is $\mathbf{4i + 3j}$. This is normal to the plane $\mathbf{4x + 3y + 0z = 0}$. (The right side could be any number.) One line in this plane is $\mathbf{4x + 3y = 0, z = 0}$. (A point that satisfies those two equations also satisfies the plane equation.)

10 The line is $(x,y,z) = (3,1,-2) + \mathbf{t}(-\mathbf{1}, -\frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{2}}{\mathbf{3}})$. Then at $t = 3$ this gives $(0,0,0)$. The speed is $\frac{\text{distance}}{\text{time}} = \frac{\sqrt{9+1+4}}{3} = \frac{\sqrt{\mathbf{14}}}{\mathbf{3}}$. For speed e^t choose $(x,y,z) = (\mathbf{3,1,-2}) + \frac{\mathbf{e^t}}{\sqrt{\mathbf{14}}}(-\mathbf{3},-\mathbf{1},\mathbf{2})$.

12 $\mathbf{x = \cos e^t, y = \sin e^t}$ has velocity $\frac{dx}{dt} = (-\sin e^t)e^t, \frac{dy}{dt} = (\cos e^t)e^t$ and speed $\sqrt{(dx/dt)^2 + (dy/dt)^2} = e^t$. The circle is complete when $e^t = 2\pi$ or $\mathbf{t = \ln 2\pi}$.

14 $x^2 + y^2 = (1+t)^2 + (2-t)^2$ is a minimum when $2(1+t) - 2(2-t) = 0$ or $4t = 2$ or $\mathbf{t = \frac{1}{2}}$. The path crosses $y = x$ when $1 + t = 2 - t$ or $t = \frac{1}{2}$ (again) at $\mathbf{x = y = \frac{3}{2}}$. The line never crosses a **parallel line like** $\mathbf{x = 2 + t, y = 2 - t}$.

16 (b)(c)(d) give the same path. Change t to $2t, -t$, and t^3, respectively. Path (a) never goes through (1,1).

18 If $x = 1 + v_1 t = 0$ and $y = 2 + v_2 t = 0$, the first gives $t = -\frac{1}{v_1}$ and then the second gives $2 - \frac{v_2}{v_1} = 0$ or $\mathbf{2v_1 - v_2 = 0}$. This line crosses the 45° line unless $v_1 = v_2$ or $\mathbf{v_1 - v_2 = 0}$. In that case $x = y$ leads to $1 = 2$ and is impossible.

20 If $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ along a path then $\frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{x^2 + y^2}) = \mathbf{0}$ and $x^2 + y^2 =$ constant.

22 If $\mathbf{a}$ is a constant vector the path must be **a straight line** (with uniform motion since $x = x_0 + x_1 t$ and $y = y_0 + v_2 t$ are the only functions with $\frac{d^2x}{dt^2} = 0 = \frac{d^2y}{dt^2}$). If the path is a straight line, $\mathbf{a}$ must be **in the same direction as the line** (*but not* necessarily constant).

24 $x = 1 + 2\cos\frac{t}{2}$ and $y = 3 + 2\sin\frac{t}{2}$. Check $(x-1)^2 + (y-3)^2 = 4$ and speed $= 1$.

26 $|\mathbf{a}| = \frac{d^2s}{dt^2}$ when the motion is **along a straight line.** On a curve there is a turning component – for example $\mathbf{x = \cos t, y = \sin t}$ has $\frac{ds}{dt} = 1$ and then $\frac{d^2s}{dt^2} = 0$ but $\mathbf{a} = -\cos t\ \mathbf{i} - \sin t\ \mathbf{j}$ is not zero.

28 $\frac{ds}{dt} = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2} = \sqrt{36 + 9 + 4} = \mathbf{7}$. The path leaves (1,2,0) when $t = 0$ and arrives at (13,8,4) when $t = 2$, so the distance is $2 \cdot 7 = \mathbf{14}$. Also $12^2 + 6^2 + 4^2 = 14^2$.

30 If the parametric equations are $\mathbf{x = \cos\theta, y = \sin\theta, z = \theta}$, the speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}$ $= \sqrt{(\sin^2\theta + \cos^2\theta)(d\theta/dt)^2 + (d\theta/dt)^2} = \sqrt{2}|d\theta/dt|$. (In Example 7 the speed was $\sqrt{2}$.) So take $\theta = \mathbf{t}/\sqrt{\mathbf{2}}$ for speed 1.

32 Given only the path $y = f(x)$, it is impossible to find the **velocity** but still possible to find the **tangent vector** (or the *slope*).

34 $x = \cos(1 - e^{-t}), y = \sin(1 - e^{-t})$ goes around the unit circle $x^2 + y^2 = 1$ with speed e^{-t}. The path starts at (1,0) when $t = 0$; it ends at $x = \cos 1, y = \sin 1$ when $t = \infty$. Thus it covers only *one radian* (because the distance is $\int (ds/dt)dt = \int e^{-t} = 1$). *Note:* The path $x = \cos e^{-t}, y = \sin e^{-t}$ is also acceptable, going from (cos 1, sin 1) backward to (1,0).

36 This is the path of a ball thrown upward: $x = 0, y = \mathbf{v_0 t} - \frac{1}{2}\mathbf{t^2}$. Take $\mathbf{v_0} = \mathbf{5}$ to return to $y = 0$ at $t = 10$.

38 The shadow on the xz plane is $\mathbf{ti} + \mathbf{t^3 k}$. The original curve has tangent direction $\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$. This is never parallel to $\mathbf{i} + \mathbf{j} + \mathbf{k}$ (along the line $x = y = z$), because $2t = 1$ and $3t^2 = 1$ happen at different times.

40 The first particle has speed 1 and arrives at $t = \frac{\pi}{2}$. The second particle arrives when $v_2 t = 1$ and $-v_1 t = 1$, so $t = \frac{1}{v_2}$ and $v_1 = -v_2$. Its speed is $\sqrt{v_1^2 + v_2^2} = \sqrt{2}v_2$. So it should have $\sqrt{\mathbf{2}}\mathbf{v_2} < \mathbf{1}$ (to go slower) and $\frac{\mathbf{1}}{\mathbf{v_2}} < \frac{\pi}{\mathbf{2}}$ (to win), OK to take $\mathbf{v_2} = \frac{\mathbf{2}}{\mathbf{3}}$.

42 $\mathbf{v} \times \mathbf{w}$ is perpendicular to both lines, so the distance between lines is the length of the projection of $\mathbf{u} = Q - P$ onto $\mathbf{v} \times \mathbf{w}$. The formula for the distance is $\frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{|\mathbf{v} \times \mathbf{w}|}$.

44 Minimize $(1+t-9)^2 + (1+2t-4)^2 + (3+2t-5)^2$ by taking the t derivative: $2(t-8) + 2(2t-3)2 + 2(2t-2)2 = 0$ or $18t = 36$. Thus $\mathbf{t = 2}$ and the closest point on the line is $\mathbf{x = 3, y = 5, z = 7}$. Its distance from (9, 4, 5) is $\sqrt{6^2 + 1^2 + 2^2} = \sqrt{41}$.

46 Time in hours, length in meters. The angle of the minute hand is $\frac{\pi}{2} - 2\pi t$ (at $t = 1$ it is back to vertical). The snail is at radius t, so $x = t\cos(\frac{\pi}{2} - 2\pi t)$ and $y = t\sin(\frac{\pi}{2} - 2\pi t)$. Simpler formulas are $\mathbf{x = t \sin 2\pi\, t}$ and $\mathbf{y = t \cos 2\pi\, t}$.

12.2 Plane Motion: Projectiles and Cycloids (page 457)

A projectile starts with speed v_0 and angle α. At time t its velocity is $dx/dt = \mathbf{v_0 \cos \alpha}, dy/dt = \mathbf{v_0 \sin \alpha - gt}$ (the downward acceleration is g). Starting from (0,0), the position at time t is $x = \mathbf{v_0 \cos \alpha\, t}, y = \mathbf{v_0 \sin \alpha\, t} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{gt^2}$. The flight time back to $y = 0$ is $T = \mathbf{2v_0(\sin \alpha)/g}$. At that time the horizontal range is $R = \mathbf{(v_0^2 \sin 2\alpha)/g}$. The flight path is a **parabola**.

The three quantities $v_0, \alpha, \mathbf{t}$ determine the projectile's motion. Knowing v_0 and the position of the target, we cannot solve for α. Knowing α and the position of the target, we can solve for v_0.

A **cycloid** is traced out by a point on a rolling circle. If the radius is a and the turning angle is θ, the center of the circle is at $x = \mathbf{a}\theta, y = \mathbf{a}$. The point is at $x = \mathbf{a(\theta - \sin \theta)}, y = \mathbf{a(1 - \cos \theta)}$, starting from (0,0). It travels a distance $\mathbf{3\pi^2}$ in a full turn of the circle. The curve has a **cusp** at the end of every turn. An upside-down cycloid gives the **fastest** slide between two points.

1 (a) $T = 16/g\,\text{sec}, R = 144\sqrt{3}/g\,\text{ft}, Y = 32/g\,\text{ft}$ **3** $x = 1.2$ or 33.5
5 $y = x - \frac{1}{2}x^2 = 0$ at $x = 2$; $y = x\tan x - \frac{g}{2}(\frac{x}{v_0 \cos \alpha})^2 = 0$ at $x = R$ **7** $x = v_0\sqrt{\frac{2h}{g}}$
9 $v_0 \approx 11.3, \tan \alpha \approx 4.4$ **11** $v_0 = \sqrt{gR} = \sqrt{980}$ m/sec; larger **13** $v_0^2/2g = 40$ meters
15 Multiply R and H by 4; $dR = 2v_0^2 \cos 2\alpha d\alpha/g, dH = v_0^2 \sin \alpha \cos\, \alpha\, d\alpha/g$
17 $t = \frac{12\sqrt{2}}{10}$ sec; $y = 12 - \frac{144g}{100} \approx -2.1$ m; + 2.1m **19** $\mathbf{T} = \frac{(1-\cos\theta)\mathbf{i} + \sin\theta\mathbf{j}}{\sqrt{2-2\cos\theta}}$
21 Top of circle **25** $ca(1 - \cos\theta), ca\sin\theta; \theta = \pi, \frac{\pi}{2}$ **27** After $\theta = \pi : x = \pi a + v_0 t$ and $y = 2a - \frac{1}{2}gt^2$ **29** 2; 3
31 $\frac{64\pi a^2}{3}; 5\pi^2 a^3$ **33** $x = \cos\theta + \theta\sin\theta, y = \sin\theta - \theta\cos\theta$ **35** $(a = 4)\ 6\pi$

37 $y = 2\sin\theta - \sin 2\theta = 2\sin\theta(1-\cos\theta)$; $x^2+y^2 = 4(1-\cos\theta)^2$; $r = 2(1-\cos\theta)$

2 $T = \frac{2v_0 \sin\alpha}{g}$ gives $1 = \frac{2(32)\sin\alpha}{32}$ or $\sin\alpha = \frac{1}{2}$ and $\alpha = 30°$; the range is $R = \frac{v_0^2 \sin 2\alpha}{g} = 32(\frac{\sqrt{3}}{2}) = 16\sqrt{3}$ ft.

4 $\mathbf{v}(0) = 3\mathbf{i} + 3\mathbf{j}$ has angle $\alpha = \frac{\pi}{4}$ and magnitude $\mathbf{v_0} = 3\sqrt{2}$. Then $v(t) = 3\mathbf{i} + (3-gt)\mathbf{j}$, $v(1) = 3\mathbf{i} - 29\mathbf{j}$ (in feet), $v(2) = 3\mathbf{i} - 26\mathbf{j}$. The position vector is $\mathbf{R}(t) = 3t\mathbf{i} + (3t - \frac{1}{2}gt^2)\mathbf{j}$, with $\mathbf{R}(1) = 3\mathbf{i} - 10\mathbf{j}$ and $\mathbf{R}(2) = 6\mathbf{i} - 58\mathbf{j}$.

6 If the maximum height is $\frac{(v_0 \sin\alpha)^2}{2g} = 6$ meters, then $\sin^2\alpha = \frac{12(9.8)}{30^2} \approx .13$ gives $\alpha \approx .37$ or $21°$.

8 The path $x = v_0(\cos\alpha)t$, $y = v_0(\sin\alpha)t - \frac{1}{2}gt^2$ reaches $y = -h$ when $\frac{1}{2}gT^2 - v_0(\sin\alpha)T - h = 0$. This quadratic equation gives $T = \frac{v_0 \sin\alpha + \sqrt{v_0^2 \sin^2\alpha + 2h}}{g}$. At that time $x = v_0(\cos\alpha)T$. The angle to maximize x has $\frac{dx}{d\alpha} = \frac{d}{d\alpha} v_0(\cos\alpha)T = 0$.

10 Substitute into $(gx/v_0)^2 + 2gy = g^2t^2\cos^2\alpha + 2gv_0 t\sin\alpha - t^2 = 2gv_0 t \sin\alpha - g^2t^2\sin^2\alpha$. This is less than v_0^2 because $(\mathbf{v_0} - \mathbf{g\, t \sin\alpha})^2 \geq 0$. For $y = H$ the largest x is when equality holds: $v_0^2 = (gx/v_0)^2 + 2gH$ or $\mathbf{x} = \sqrt{\mathbf{v_0^2} - \mathbf{2gH}(\frac{\mathbf{v_0}}{\mathbf{g}})}$. If $2gH$ is larger than v_0, the height H can't be reached.

12 T is in seconds and R is in meters if v_0 is in **meters per second** and g is in $\mathbf{m/sec^2}$.

14 time $= \frac{\text{distance}}{\text{speed}} = \frac{60 \text{ feet}}{100 \text{ miles/hour}} = \frac{60 \text{ feet}}{100(5280) \text{ feet/hour}} = .41$ seconds. In that time the fall $\frac{1}{2}gt^2$ is 2.7 feet.

16 The speed is the square root of $(v_0\cos\alpha)^2 + (v_0 \sin\alpha - gt)^2 = v_0^2 - 2v_0(\sin\alpha)gt + g^2t^2$. The derivative is $-2v_0(\sin\alpha)g + 2g^2t = 0$ when $t = \frac{v_0(\sin\alpha)}{g}$. This is the top of the path, where the speed is a minimum. The maximum speed must be v_0 (at $t = 0$ and also at the endpoint $t = \frac{2v_0(\sin\alpha)}{g}$).

18 For a large v_0 and a given R= distance to hole, there will be *two* angles that satisfy $R = \frac{v_0^2 \sin 2\alpha}{g}$. The low trajectory (small α) would encounter less air resistance than the high trajectory (large α).

20 $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$ becomes $\frac{0}{0}$ at $\theta = 0$, so use **l'Hôpital's Rule**: The ratio of derivatives is $\frac{\cos\theta}{\sin\theta}$ which becomes infinite. $\frac{\sin\theta}{1-\cos\theta} \approx \frac{\theta}{\theta^2/2} = \frac{2}{\theta}$ equals **20** at $\theta = \frac{1}{10}$ and **−20** at $\theta = -\frac{1}{10}$. The slope is 1 when $\sin\theta = 1 - \cos\theta$ which happens at $\theta = \frac{\pi}{2}$.

22 Change Figure 12.6b so the line from C to the new P' has length d not a. The components are $-d\sin\theta$ and $-d\cos\theta$. Then $x = a\theta - d\sin\theta$ and $\mathbf{y = a - d\cos\theta}$.

24 $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$ by Problem 20. The θ derivative is $\frac{(1-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1-\cos\theta)^2} = \frac{\cos\theta - 1}{(1-\cos\theta)^2} = \frac{-1}{1-\cos\theta}$. This is $\frac{d}{d\theta}(\frac{dy}{dx}) = \frac{d^2y}{dx^2}\frac{dx}{d\theta}$. So divide by $\frac{dx}{d\theta} = 1 - \cos\theta$ to find $\frac{d^2y}{dx^2} = \frac{-1}{(1-\cos\theta)^2}$. This is negative and the cycloid is convex **down**.

26 The curves $x = a\cos\theta + b\sin\theta$, $y = c\cos\theta + d\sin\theta$ are closed because at $\theta = 2\pi$ they come back to the starting point and repeat.

32 For $c = 1$ the curve is $x = 2\cos\theta$, $y = 0$ which is a horizontal line segment on the axis from $x = -2$ to $x = 2$. As in Problem 23, when a circle of radius 1 rolls inside a circle of radius 2, one point goes across in a straight line.

34 The arc of the big circle in the astroid figure has length 4θ (radius times central angle) so the arc of the small circle is also 4θ. Its radius is 1, so the indicated angle of 3θ plus the angle θ above it give the correct angle 4θ.

To get from O to P go along the radius to $(3\cos\theta, 3\sin\theta)$, then down the short radius to $(x, y) = (3\cos\theta + \cos 3\theta, 3\sin\theta - \sin 3\theta)$. Use $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = -4\sin^3\theta + 3\sin\theta$ to convert to $x = 4\cos^3\theta$ and $y = 4\sin^3\theta$.

36 The biggest triangle in the "Witch figure" has side $2a$ opposite an angle θ at the point A.

So $\frac{2a}{\text{distance across}} = \tan\theta$ and $x =$ distance across $= \frac{2a}{\tan\theta} = 2a\cot\theta$. The length OB is $2a\sin\theta$ (from the polar equation of a circle in Figure 9.2c, or from plane geometry). Then the height of B is $(OB)(\sin\theta) = 2a\sin^2\theta$. The identity $1+\cot^2\theta = \csc^2\theta$ gives $\mathbf{1 + (\frac{x}{2a})^2 = \frac{2a}{y}}$.

38 On the line $x = \frac{\pi}{2}y$ the distance is $ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{(\pi/2)^2+1}\,dy$. The last step in equation (5) integrates $\frac{\text{constant}}{\sqrt{y}}$ to give $\frac{\sqrt{\pi^2+4}}{2\sqrt{2g}}[2\sqrt{y}]_0^{2a} = \sqrt{\pi^2+4}\frac{2\sqrt{2a}}{2\sqrt{2g}} = \sqrt{\pi^2+4}\sqrt{\frac{a}{g}}$.

40 I have read (but don't believe) that the rolling circle jumps as the weight descends.

12.3 Curvature and Normal Vector (page 463)

The curvature tells how fast the curve **turns**. For a circle of radius a, the direction changes by 2π in a distance $\mathbf{2\pi a}$, so $\kappa = \mathbf{1/a}$. For a plane curve $y = f(x)$ the formula is $\kappa = |y''|/(\mathbf{1 + (y')^2})^{\mathbf{3/2}}$. The curvature of $y = \sin x$ is $|\mathbf{\sin x}|/(\mathbf{1+\cos^2 x})^{\mathbf{3/2}}$. At a point where $y'' = 0$ (an **inflection point**) the curve is momentarily straight and $\kappa =$ **zero**. For a space curve $\kappa = |\mathbf{v}\times\mathbf{a}|/|\mathbf{v}|^{\mathbf{3}}$.

The normal vector $\mathbf{N}$ is perpendicular to **the curve (and therefore to v and T)**. It is a **unit** vector along the derivative of $\mathbf{T}$, so $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$. For motion around a circle $\mathbf{N}$ points **inward**. Up a helix $\mathbf{N}$ also points **inward**. Moving at unit speed on any curve, the time t is the same as the **distance** s. Then $|\mathbf{v}| = \mathbf{1}$ and $d^2s/dt^2 = \mathbf{0}$ and $\mathbf{a}$ is in the direction of $\mathbf{N}$.

Acceleration equals $\mathbf{d^2s/dt^2\,T} + \kappa|\mathbf{v}|^{\mathbf{2}}\,\mathbf{N}$. At unit speed around a unit circle, those components are **zero** and **one**. An astronaut who spins once a second in a radius of one meter has $|\mathbf{a}| = \omega^{\mathbf{2}} = (\mathbf{2\pi})^{\mathbf{2}}$ meters/sec^2, which is about **4g**.

1 $\frac{e^x}{(1+e^{2x})^{3/2}}$ **3** $\frac{1}{2}$ **5** 0 (line) **7** $\frac{2+t^2}{(1+t^2)^{3/2}}$ **9** $(-\sin t^2, \cos t^2); (-\cos t^2, -\sin t^2)$
11 $(\cos t, \sin t); (-\sin t, -\cos t)$ **13** $(-\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}); |\mathbf{v}| = 5, \kappa = \frac{3}{25}; \frac{5}{3}$ longer; $\tan\theta = \frac{4}{3}$
15 $\frac{1}{2\sqrt{2a}\sqrt{1-\cos\theta}}$ **17** $\kappa = \frac{3}{16}, \mathbf{N} = \mathbf{i}$ **19** $(0,0); (-3,0)$ with $\frac{1}{\kappa} = 4; (-1,2)$ with $\frac{1}{\kappa} = 2\sqrt{2}$
21 Radius $\frac{1}{\kappa}$, center $(1, \pm\sqrt{\frac{1}{\kappa^2}-1})$ for $\kappa \le 1$ **23** $\mathbf{U}\cdot\mathbf{V}'$ **25** $\frac{1}{\sqrt{2}}(\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k})$ **27** $\frac{1}{2}$
29 $\mathbf{N}$ in the plane, $\mathbf{B} = \mathbf{k}$, $\tau = 0$ **31** $\frac{d^2y/dx^2}{1+(dy/dx)^2}$ **33** $\mathbf{a} = 0\,\mathbf{T} + 5\omega^2\mathbf{N}$ **35** $\mathbf{a} = \frac{t}{\sqrt{1+t^2}}\mathbf{T} + \frac{2+t^2}{\sqrt{1+t^2}}\mathbf{N}$
37 $\mathbf{a} = \frac{4t}{\sqrt{1+4t^2}}\mathbf{T} + \frac{2}{\sqrt{1+4t^2}}\mathbf{N}$ **39** $|F^2 + 2(F')^2 - FF''|/(F^2+F'^2)^{3/2}$

2 $y = \ln x$ has $\kappa = \frac{|y''|}{(1+y'^2)^{3/2}} = \frac{1/x^2}{(1+\frac{1}{x^2})^{3/2}} = \frac{\mathbf{x}}{(\mathbf{x^2+1})^{\mathbf{3/2}}}$. Maximum of κ when its derivative is zero: $(x^2+1)^{3/2} = x\frac{3}{2}(x^2+1)^{1/2}(2x)$ or $x^2+1 = 3x^2$ or $\mathbf{x^2 = \frac{1}{2}}$.

4 $x = \cos t^2, y = \sin t^2$ has $x' = -2t\sin t^2$ and $y' = 2t\cos t^2$. Then $x'' = -2\sin t^2 - 4t^2\cos t^2$ and $y'' = 2\cos t^2 - 4t^2\sin t^2$. Therefore $\kappa = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}} = \frac{8t^3(\sin t^2)^2+8t^3(\cos t^2)^2}{(4t^2(\sin t^2)^2+4t^2(\cos t^2)^2)^{3/2}} = \frac{8t^3}{(4t^2)^{3/2}} = \mathbf{1}$. Reason: κ depends only on the path (not the speed) and this path is a **unit circle**.

6 $x = \cos^3 t$ has $x' = -3\cos^2 t\sin t$ and $x'' = -3\cos^3 t + 6\cos t\sin^2 t$; $y = \sin^3 t$ has $y' = 3\sin^2 t\cos t$ and $y'' = -3\sin^3 t + 6\sin t\cos^2 t$. Then $x'y'' - y'x'' = -9\cos^2 t\sin^4 t - 9\sin^2 t\cos^4 t = -9\cos^2 t\sin^2 t$.

Also $(x')^2 + (y')^2 = 9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t = 9\cos^2 t\sin^2 t$. The $\frac{3}{2}$ power is $27\cos^3 t\sin^3 t$ and division leaves $\kappa = \frac{1}{3\cos t\sin t}$.

8 $x = t, y = \ln\cos t$ has $x' = 1, x'' = 0, y' = \tan t, y'' = \sec^2 t$. Then $\kappa = \frac{\sec^2 t}{(1+\tan^2 t)^{3/2}} = \frac{\sec^2 t}{\sec^3 t} = \mathbf{\cos t}$.

10 Problem 6 has $\mathbf{v} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} = -3\cos^2 t\sin t\;\mathbf{i} + 3\sin^2 t\cos t\;\mathbf{j} = 3\;\cos t\sin t$ times a unit vector $-\cos t\;\mathbf{i} + \sin t\;\mathbf{j}$. Perpendicular to $\mathbf{T}$ is the normal $\mathbf{N} = \sin t\;\mathbf{i} + \cos t\;\mathbf{j}$ (also a unit vector).

12 $x' = v_0\cos\alpha, x'' = 0, y' = v_0\sin\alpha - gt, y'' = -g$. Therefore $|\mathbf{v}|^2 = v_0^2(\cos^2\alpha + \sin^2\alpha) - 2v_0(\sin\alpha)gt + g^2t^2$ or $|\mathbf{v}|^2 = \mathbf{v_0^2 - 2v_0(\sin\alpha)gt + g^2t^2}$. Also $\kappa = \frac{|x'y''-y'x''|}{|v|^3} = \frac{gv_0\cos\alpha}{|v|^3}$. (Note: $\kappa = \frac{g\cos\alpha}{v_0^2}$ at $t = 0$.)

14 When $\kappa = 0$ the path is a **straight line**. This happens when $\mathbf{v}$ and $\mathbf{a}$ are **parallel**. Then $\mathbf{v}\times\mathbf{a} = \mathbf{0}$.

16 In $\kappa = \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}$, doubling x and y **multiplies** κ **by** $\mathbf{\frac{4}{4^{3/2}} = \frac{1}{2}}$. (Less curvature for wider curve.) The velocity has a factor 2 but the unit vectors $\mathbf{T}$ and $\mathbf{N}$ are **unchanged**.

18 Using equation (8), $\mathbf{v}\times\mathbf{a} = |\mathbf{v}|\mathbf{T}\times(\frac{d^2s}{dt^2}\mathbf{T} + \kappa(\frac{ds}{dt})^2\mathbf{N}) = \kappa|\mathbf{v}|^3\mathbf{T}\times\mathbf{N}$ because $\mathbf{T}\times\mathbf{T} = \mathbf{0}$ and $|\mathbf{v}|$ is the same as $|\frac{ds}{dt}|$. Since $|\mathbf{T}\times\mathbf{N}| = 1$ this gives $|\mathbf{v}\times\mathbf{a}| = \kappa|\mathbf{v}|^3$ or $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$.

20 $\mathbf{v}$ and $|\mathbf{v}|$ and $\mathbf{a}$ depend on the speed along the curve; $\mathbf{T}$ and s and κ and $\mathbf{N}$ and $\mathbf{B}$ depend only on the path (the shape of the curve).

22 The parabola through the three points is $y = x^2 - 2x$ which has a constant second derivative $\frac{d^2y}{dx^2} = 2$. The circle through the three points has radius $= 1$ and $\kappa = \frac{1}{\textbf{radius}} = 1$. These are the smallest possible (Proof?)

24 If $\mathbf{v}$ is perpendicular to $\mathbf{a}$, then $\frac{d}{dt}\mathbf{v}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{v} = 0 + 0 = 0$. So $\mathbf{v}\cdot\mathbf{v} =$ constant or $|\mathbf{v}|^2 =$ constant. The path does *not* have to be a circle, as long as the speed is constant. Example: helix as in Section 12.1.

26 $\mathbf{B}\cdot\mathbf{T} = 0$ gives $\mathbf{B}'\cdot\mathbf{T} + \mathbf{B}\cdot\mathbf{T}' = 0$ and thus $\mathbf{B}'\cdot\mathbf{T} = 0$ (since $\mathbf{B}\cdot\mathbf{T}' = \mathbf{B}\cdot\mathbf{N} = 0$ by construction). Also $\mathbf{B}\cdot\mathbf{B} = 1$ gives $\mathbf{B}'\cdot\mathbf{B} = 0$. So $\mathbf{B}'$ must be in the direction of $\mathbf{N}$.

28 The curve $(1, t, t^2)$ has $\mathbf{v} = (0, 1, 2t)$. So $\mathbf{T}$ is a combination of $\mathbf{j}$ and $\mathbf{k}$, and so are $d\mathbf{T}/dt$ and $\mathbf{N}$. The perpendicular direction $\mathbf{B} = \mathbf{T}\times\mathbf{N}$ must be $\mathbf{i}$.

30 The product rule for $\mathbf{N} = -\mathbf{T}\times\mathbf{B}$ gives $\frac{d\mathbf{N}}{ds} = -\mathbf{T}\times\frac{d\mathbf{B}}{ds} - \frac{d\mathbf{T}}{ds}\times\mathbf{B} = \mathbf{T}\times\tau\mathbf{N} - \kappa\mathbf{N}\times\mathbf{B} = \tau\mathbf{B} - \kappa\mathbf{T}$.

32 $\mathbf{T} = \cos\theta\;\mathbf{i} + \sin\theta\;\mathbf{j}$ gives $\frac{d\mathbf{T}}{d\theta} = -\sin\theta\;\mathbf{i} + \cos\theta\;\mathbf{j}$ so $|\frac{d\mathbf{T}}{d\theta}| = 1$. Then $\kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\mathbf{T}}{d\theta}||\frac{d\theta}{ds}| = |\frac{d\theta}{ds}|$. **Curvature is rate of change of slope of path.**

34 $(x, y, z) = (1, 1, 1) + t(1, 2, 3)$ has $\mathbf{v} = (1,2,3)$ and $\frac{ds}{dt} = \frac{d^2s}{dt^2} = 0$. Then $\kappa = 0$. So $\mathbf{a} = \mathbf{0}$. This is uniform motion in a straight line.

36 $x' = e^t(\cos t - \sin t), y' = e^t(\sin t + \cos t), x'' = e^t(\cos t - \sin t - \sin t - \cos t), y'' = e^t(\sin t + \cos t + \cos t - \sin t)$. Then $(\frac{ds}{dt})^2 = (x')^2 + (y')^2 = e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t) = \mathbf{2e^{2t}}$. Thus $\frac{ds}{dt} = \sqrt{2}e^t$ and $\frac{d^2s}{dt^2} = \sqrt{2}e^t$. Also $x'y'' - y'x'' = e^{2t}[(\cos t - \sin t)(2\cos t) - (\sin t + \cos t)(-2\sin t)] = \mathbf{2e^{2t}}$. So $\kappa = \frac{1}{\sqrt{2}e^t}$ by equation (5). Equation (8) is $\mathbf{a} = \sqrt{2}e^t\mathbf{T} + \sqrt{2}e^t\mathbf{N}$.

38 The spiral has $\mathbf{R} = (e^t\cos t, e^t\sin t)$ and from Problem 36, $\mathbf{a} = (x'', y'') = (-2\sin t\;e^t, 2\cos t\;e^t)$. Since $R\cdot\mathbf{a} = 0$, the angle is $90°$.

12.4 Polar Coordinates and Planetary Motion (page 468)

A central force points toward the **origin**. Then $\mathbf{R}\times d^2\mathbf{R}/dt^2 = \mathbf{0}$ because **these vectors are parallel.**

Therefore $\mathbf{R} \times d\mathbf{R}/dt$ is a **constant** (called **H**).

In polar coordinates, the outward unit vector is $\mathbf{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Rotated by $90°$ this becomes $\mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$. The position vector $\mathbf{R}$ is the distance r times $\mathbf{u_r}$. The velocity $\mathbf{v} = d\mathbf{R}/dt$ is $(\mathbf{dr/dt})\mathbf{u}_r + (\mathbf{r}\,\mathbf{d\theta/dt})\mathbf{u}_\theta$. For steady motion around the circle $r = 5$ with $\theta = 4t$, $\mathbf{v}$ is $-\mathbf{20}\sin 4\mathbf{t}\,\mathbf{i} + \mathbf{20}\cos 4\mathbf{t}\,\mathbf{j}$ and $|\mathbf{v}|$ is $\mathbf{20}$ and $\mathbf{a}$ is $-\mathbf{80}\cos 4\mathbf{t}\,\mathbf{i} - \mathbf{80}\sin 4\mathbf{t}\,\mathbf{j}$.

For motion under a circular force, r^2 times $\mathbf{d\theta/dt}$ is constant. Dividing by 2 gives Kepler's second law $dA/dt = \frac{1}{2}\mathbf{r^2 d\theta/dt} =$ **constant**. The first law says that the orbit is an **ellipse** with the sun at a **focus**. The polar equation for a conic section is $\mathbf{1/r} = C - D\cos\theta$. Using $\mathbf{F} = m\mathbf{a}$ we found $q_{\theta\theta} + \mathbf{q} = C$. So the path is a conic section; it must be an ellipse because **planets come around again**. The properties of an ellipse lead to the period $T = \mathbf{2\pi a^{3/2}/\sqrt{GM}}$, which is Kepler's third law.

1 $\mathbf{j}, -\mathbf{i}; \mathbf{i}+\mathbf{j} = \mathbf{u}_r - \mathbf{u}_\theta$ **3** $(2,-1); (1,2)$ **5** $\mathbf{v} = 3e^3(\mathbf{u}_r + \mathbf{u}_\theta) = 3e^3(\cos 3 - \sin 3)\mathbf{i} + 3e^3(\sin 3 + \cos 3)\mathbf{j}$
7 $\mathbf{v} = -20\sin 5t\,\mathbf{i} + 20\cos 5t\,\mathbf{j} = 20\,\mathbf{T} = 20\,\mathbf{u}_\theta; \mathbf{a} = -100\cos 5t\,\mathbf{i} - 100\sin 5t\,\mathbf{j} = 100\,\mathbf{N} = -100\,\mathbf{u}_r$
9 $r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0 = \frac{1}{r}\frac{d}{dt}(r^2\frac{d\theta}{dt})$ **11** $\frac{d\theta}{dt} = .0004$ radians/sec; $h = r^2\frac{d\theta}{dt} = 40{,}000$
13 $m\mathbf{R}\times\mathbf{a}$; torque **15** $T^{2/3}(GM/4\pi^2)^{1/3}$ **17** $4\pi^2a^3/T^2G$ **19** $\frac{4\pi^2(150)^3 10^{27}}{(365\frac{1}{4})^2(24)^2(3600)^2(6.67)10^{-11}}$ kg
23 Use Problem **15** **25** $a + c = \frac{1}{C-D}, a - c = \frac{1}{C+D}$, solve for C, D
27 Kepler measures area from focus (sun) **29** Line; $x = 1$
31 The path of a quark is $r^2(A + B\cos^2\theta - B\sin^2\theta) = 1$. Substitute x for $r\cos\theta$, y for $r\sin\theta$, and $x^2 + y^2$ for r^2 to find $(A+B)x^2 + (A-B)y^2 = 1$. This is an ellipse centered at the origin. (We know $A > B$ because $A + B\cos 2\theta$ must be positive in the original equation).
33 $r = 20 - 2t, \theta = \frac{2\pi t}{10}, \mathbf{v} = -2\mathbf{u}_r + (20-2t)\frac{2\pi}{10}\mathbf{u}_\theta; \mathbf{a} = (2t-20)(\frac{2\pi}{10})^2\mathbf{u}_r - 4(\frac{2\pi}{10})\mathbf{u}_\theta; \int_0^{10}|\mathbf{v}|dt$

2 The point (3,3) is at $\theta = \frac{\pi}{4}$. So $\mathbf{u_r} = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$ and $\mathbf{u}_\theta = \frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$. If $\mathbf{v} = \mathbf{i}+\mathbf{j}$ then $\mathbf{v} = \sqrt{2}\mathbf{u_r}$. This is the velocity when $\frac{dr}{dt} = \sqrt{2}$ and $\frac{d\theta}{dt} = 0$. (Better question: If $\mathbf{R} = 3\mathbf{i} + 3\mathbf{j}$ then $\mathbf{R} =$ ____ $\mathbf{u_r}$. Answer $r = \sqrt{18}$.)
4 $r = 1 - \cos\theta$ has $\frac{dr}{dt} = \sin\theta\frac{d\theta}{dt} = 2\sin\theta$. Then $\mathbf{v} = \mathbf{2}\sin\theta\,\mathbf{u_r} + \mathbf{2}(\mathbf{1} - \cos\theta)\mathbf{u}_\theta$. The cardioid is covered as θ goes from 0 to 2π. With $\frac{d\theta}{dt} = 2$ the time required is π.
6 The path $r = 1, \theta = \sin t$ goes along the unit circle from $\theta = 0$ to $\theta = 1$ radian, then backward to $\theta = -1$ radian, and oscillates on this arc. The velocity from equation (5) is $\mathbf{v} = r\frac{d\theta}{dt}\mathbf{u}_\theta = \cos t\,\mathbf{u}_\theta$; the acceleration is $\mathbf{a} = -\cos^2 t\,\mathbf{u_r} - \sin t\,\mathbf{u}_\theta$: part radial from turning, part tangential from change of speed. $\mathbf{v} = \mathbf{0}$ when $\cos t = 0$ (top and bottom of arc: $\theta = 1$ or -1).
8 The distance $r\theta$ around the circle is the integral of the speed $8t$: thus $4\theta = 4t^2$ and $\theta = t^2$. The circle is complete at $\mathbf{t} = \sqrt{2\pi}$. At that time $\mathbf{v} = r\frac{d\theta}{dt}\mathbf{u}_\theta = 4(2\sqrt{2\pi})\mathbf{j}$ and $\mathbf{a} = -\mathbf{4}(\mathbf{8}\pi)\mathbf{i} + \mathbf{4}(\mathbf{2})\mathbf{j}$.
10 The line $x = 1$ is $\mathbf{r}\cos\theta = \mathbf{1}$ or $r = \sec\theta$. Integrating $r^2\frac{d\theta}{dt} = \sec^2\theta\frac{d\theta}{dt} = 2$ gives $\tan\theta = 2t$. The point (1,1) at $\theta = \frac{\pi}{4}$ is reached when $\tan\theta = 1 = 2t$; then $\mathbf{t} = \frac{1}{2}$.
12 Since $\mathbf{u_r}$ has constant length, its derivatives are **perpendicular** to itself. In fact $\frac{d\mathbf{u}_r}{dr} = 0$ and $\frac{d\mathbf{u_r}}{d\theta} = \mathbf{u}_\theta$.
14 $R = re^{i\theta}$ has $\frac{d^2R}{dt^2} = \frac{d^2r}{dt^2}e^{i\theta} + 2\frac{dr}{dt}(ie^{i\theta}\frac{d\theta}{dt}) + ir\frac{d^2\theta}{dt^2}e^{i\theta} + i^2r(\frac{d\theta}{dt})^2e^{i\theta}$. (Note repeated term gives factor 2.) The coefficient of $e^{i\theta}$ is $\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2$. The coefficient of $ie^{i\theta}$ is $2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$. These are the $\mathbf{u_r}$

and $\mathbf{u}_\theta$ components of $\mathbf{a}$.

16 The period of a satellite above New York is **1 day = 86,400 seconds.** Then $86{,}400 = \frac{2\pi}{\sqrt{GM}} a^{3/2}$ gives $a = 4.2 \cdot 10^7$ meters = 420,000 km.

18 The period of the moon reveals the mass of the earth: 28 days $\cdot 86400 \frac{\text{sec}}{\text{day}} = \frac{2\pi}{\sqrt{GM}}(380{,}000)^{3/2}$ gives $M = 5.54 \cdot 10^{24}$ kg. Remember to change 380,000 km to meters.

20 (a) **False:** The paths are conics but they could be hyperbolas and possibly parabolas.

(b) **True:** A circle has r = constant and $r^2 \frac{d\theta}{dt}$ = constant so $\frac{d\theta}{dt}$ = constant.

(c) **False:** The central force might not be proportional to $\frac{1}{r^2}$.

22 $T = \frac{2\pi}{\sqrt{GM}}(9000)^{3/2} \approx .268$ seconds.

24 $1 = Cr - Dx$ is $1 + Dx = Cr$ or $\mathbf{1 + 2Dx + D^2x^2 = C^2(x^2 + y^2)}$. Then $(C^2 - D^2)x^2 + C^2y^2 - 2Dx = 1$.

26 Substitute $x = -c, y = \frac{b^2}{a}$ and use $c^2 = a^2 - b^2$. Then $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{c^2}{a^2} + \frac{b^4/a^2}{b^2} = \frac{c^2+b^2}{a^2} = 1$.

28 If the force is $\mathbf{F} = -ma(r)\mathbf{u}_r$, the left side of equation (11) becomes $-a(r)$. Gravity has $\mathbf{a(r)} = \frac{\mathbf{GM}}{\mathbf{r^2}}$.

30 Multiply $q_{\theta\theta} + q = \frac{1}{q^3}$ by q_θ and integrate: $\frac{1}{2}q_\theta^2 + \frac{1}{2}q^2 = \int \frac{q_\theta}{q^3} d\theta = \frac{-\mathbf{1}}{\mathbf{2q^2}} + C$. Substituting $u = q^2$ and $u_\theta = 2qq_\theta$ (or $q_\theta^2 = \frac{u_\theta^2}{4q^2} = \frac{u_\theta^2}{4u}$) gives $\frac{u_\theta^2}{8u} + \frac{u}{2} = \frac{-1}{2u} + C$ or $u_\theta^2 = -4u^2 + 8uC - 4$. Integrate $\frac{du}{\sqrt{-4u^2+8uC-4}} = d\theta$ which is inside the front cover to find $\theta + c = \frac{1}{2}\sin^{-1}\frac{u-C}{\sqrt{C^2-1}}$.
Then $\frac{1}{r^2} = u = C + \sqrt{C^2-1}\sin(2\theta + c)$.

32 $T = \frac{2\pi}{\sqrt{GM}}(1.6 \cdot 10^9)^{3/2} \approx 71$ years. So the comet will return in the year $1986 + 71 = 2057$.

34 First derivative: $\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{C - D\cos\theta}\right) = \frac{-D\sin\theta \frac{d\theta}{dt}}{(C-D\cos\theta)^2} = -D\sin\theta\, r^2\frac{d\theta}{dt} = -Dh\sin\theta$.
Next derivative: $\frac{d^2r}{dt^2} = -Dh\cos\theta\frac{d\theta}{dt} = \frac{-Dh^2\cos\theta}{r^2}$. But $C - D\cos\theta = \frac{1}{r}$ so $-D\cos\theta = (\frac{1}{r} - C)$.
The acceleration terms $\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2$ combine into $(\frac{1}{r} - C)\frac{h^2}{r^2} - \frac{h^2}{r^3} = -C\frac{h^2}{r^2}$. Conclusion by Newton:
The elliptical orbit $r = \frac{1}{C - D\cos\theta}$ requires acceleration $= \frac{\text{constant}}{r^2}$: *the inverse square law.*