CHAPTER 10 INFINITE SERIES

10.1 The Geometric Series (page 373)

The geometric series $1+x+x^2+\cdots$ adds up to $\mathbf{1/(1-x)}$. It converges provided $|x| < \mathbf{1}$. The sum of n terms is $\mathbf{(1-x^n)/(1-x)}$. The derivatives of the series match the derivatives of $1/(1-x)$ at the point $x = \mathbf{0}$, where the nth derivative is **n!** The decimal 1.111... is the geometric series at $x = \mathbf{.1}$ and equals the fraction **10/9**. The decimal .666... multiplies this by **.6**. The decimal .999... is the same as **1**.

The derivative of the geometric series is $\mathbf{1/(1-x)^2 = 1 + 2x + 3x^2} + \cdots$. This also comes from squaring the **geometric** series. By choosing $x = .01$, the decimal 1.02030405 is close to $\mathbf{(100/99)^2}$. The differential equation $dy/dx = y^2$ is solved by the geometric series, going term by term starting from $y(0) = \mathbf{1}$.

The integral of the geometric series is $\mathbf{-\ln(1-x) = x + x^2/2} + \cdots$. At $x = 1$ this becomes the **harmonic** series, which diverges. At $x = \mathbf{\frac{1}{2}}$ we find $\ln 2 = \mathbf{\frac{1}{2} + (\frac{1}{2})^2/2 + (\frac{1}{2})^3/3} + \cdots$. The change from x to $-x$ produces the series $1/(1+x) = \mathbf{1 - x + x^2 - x^3} + \cdots$ and $\ln(1+x) = \mathbf{x - x^2/2 + x^3/3} \cdots$.

In the geometric series, changing to x^2 or $-x^2$ gives $1/(1-x^2) = \mathbf{1 + x^2 + x^4} + \cdots$ and $1/(1+x^2) = \mathbf{1 - x^2 + x^4} - \cdots$. Integrating the last one yields $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \cdots = \mathbf{\tan^{-1} x}$. The angle whose tangent is $x = 1$ is $\tan^{-1} 1 = \mathbf{1 - \frac{1}{3} + \frac{1}{5}} - \cdots$. Then substituting $x = 1$ gives the series $\pi = \mathbf{4(1 - \frac{1}{3} + \frac{1}{5}} - \cdots)$.

1 Subtraction leaves $G - xG = 1$ or $G = \frac{1}{1-x}$ **3** $\frac{1}{2}; \frac{4}{5}; \frac{100}{11}; 3\frac{4}{99}$ **5** $2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 + \cdots = \frac{2}{(1-x)^3}$
7 .142857 repeats because the next step divides 7 into 1 again
9 If q (prime, not 2 or 5) divides $10^N - 10^M$ then it divides $10^{N-M} - 1$ **11** This decimal does not repeat
13 $\frac{87}{99}; \frac{123}{999}$ **15** $\frac{x}{1-x^2}$ **17** $\frac{x^3}{1-x^3}$ **19** $\frac{\ln x}{1-\ln x}$ **21** $\frac{1}{x-1}$ **23** $\tan^{-1}(\tan x) = x$
25 $(1 + x + x^2 + x^3 \cdots)(1 - x + x^2 - x^3 \cdots) = 1 + x^2 + x^4 + \cdots$
27 $2(.1234\ldots)$ is $2 \cdot \frac{1}{10} \cdot \frac{1}{(1-\frac{1}{10})^2} = \frac{20}{81}$; $1 - .0123\ldots$ is $1 - \frac{1}{100}\frac{1}{(1-\frac{1}{10})^2} = \frac{80}{81}$ **29** $\frac{2}{3}\frac{1}{1-\frac{1}{3}} = 1$
31 $-\ln(1 - .1) = -\ln .9$ **33** $\frac{1}{2} \ln \frac{1.1}{.9}$ **35** $(n+1)!$ **37** $y = \frac{b}{1-bx}$
39 All products like $a_1 b_2$ are missed; $(1+1)(1+1) \neq 1+1$ **41** Take $x = \frac{1}{2}$ in (13): $\ln 3 = 1.0986$
43 In 3 seconds the ball goes 78 feet **45** $\tan z = \frac{2}{3}$; (18) is slower with $x = \frac{2}{3}$

2 Distances down and up: $10 + 6 + 6 + 6 \cdot \frac{3}{5} + 6 \cdot \frac{3}{5} + \cdots = 10 + 2(6)\frac{1}{1-\frac{3}{5}} = \mathbf{40}$ **feet**
4 $1 + (1-x) + (1-x)^2 + \cdots = \frac{1}{1-(1-x)} = \frac{1}{x}$; integration gives $\ln x = x - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \cdots + C$ and at $x = 1$ we find $C = -1$. Therefore $\mathbf{\ln x = -[(1-x) + \frac{(1-x)^2}{2} + \frac{(1-x)^3}{3}} + \cdots]$. At $x = 0$ this is $-\infty = -\infty$.
6 Multiplying $(1 - x + x^2 - \cdots)$ times $(1 + x + x^2 + \cdots)$ term by term, the odd powers disappear and one of each even power survives. The product is $1 + x^2 + x^4 + \cdots$ which is $\frac{1}{1-x^2} = \frac{1}{1+x}\frac{1}{1-x}$, the product of the two series.
8 $\frac{1}{13} = .076923076923\cdots$ so that $c = 76923$ and $N = 6$ and $n = 6$ (repeat after 6 digits starting immediately).
10 The decimal $.010010001\cdots$ is not repeating because the number of zeros increases. So it cannot be a fraction.
12 The number $1.065065\cdots$ equals $1 + \frac{65}{1000} + \frac{65}{(1000)^2} + \cdots$ which is $1 + \frac{65}{1000}(\frac{1}{1-\frac{1}{1000}}) = 1 + \frac{65}{999} = \mathbf{\frac{1064}{999}}$.
14 $(1+\frac{1}{10}+\frac{1}{100}+\cdots)(1+\frac{1}{10}+\frac{1}{100}+\cdots) = \mathbf{1 + \frac{2}{10} + \frac{3}{100} + \frac{4}{1000}} + \cdots$ Expressed in fractions this is $(\frac{10}{9})^2 = \frac{100}{81}$.
16 $1 - 2x + (2x)^2 - \cdots = \frac{1}{1-(-2x)} = \mathbf{\frac{1}{1+2x}}$.
18 $\frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{8}x^3 - \cdots = \frac{x}{2} - (\frac{x}{2})^2 + (\frac{x}{2})^3 - \cdots = \frac{\frac{x}{2}}{1+\frac{x}{2}} = \mathbf{\frac{x}{2+x}}$.

20 $x - 2x^2 + 3x^3 - \cdots = x(1 - 2x + 3x^2 - \cdots) =$ [change x to $-x$ in equation (5)] $= \mathbf{\frac{x}{(1+x)^2}}$.

22 $x(1 + \frac{1}{1+x} + \frac{1}{(1+x)^2} + \cdots) = x(\frac{1}{1-\frac{1}{1+x}}) = x(\frac{1+x}{1+x-1}) = \mathbf{1 + x}$.

24 $e^x + e^{2x} + e^{3x} + \cdots = e^x(1 + e^x + e^{2x} + \cdots) = \mathbf{e^x(\frac{1}{1-e^x})}$.

26 $\int(1 + x^2 + x^4 + \cdots)dx = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \mathbf{\frac{1}{2} \ln \frac{1+x}{1-x}}$ by equation (13). This is $\int \frac{1}{1-x^2}dx$ which is also $\mathbf{\tanh^{-1} x}$ ($+ C$).

28 $(1 + x + x^2)(1 + x + x^2)(1 + x + x^2) = (1 + 2x + 3x^2 + \cdots)(1 + x + x^2) = \mathbf{1 + 3x + 6x^2} + \cdots$

30 $.1 + .02 + .003 + \cdots = \frac{1}{10} + 2(\frac{1}{10})^2 + 3(\frac{1}{10})^3 + \cdots =$ by equation (5) $= \frac{1}{10}\frac{1}{(1-\frac{1}{10})^2} = \mathbf{\frac{10}{81}}$.

32 $\frac{1}{10} - \frac{1}{2}(\frac{1}{10})^2 + \frac{1}{3}(\frac{1}{10})^3 - \cdots =$ by equation (10b) $= \ln(1 + \frac{1}{10}) = \mathbf{\ln 1.1}$.

34 $1 - \frac{1}{3}(\frac{1}{3}) + \frac{1}{5}(\frac{1}{3})^2 - \cdots = \sqrt{3}[\frac{1}{\sqrt{3}} - \frac{1}{3}(\frac{1}{\sqrt{3}})^3 + \frac{1}{5}(\frac{1}{\sqrt{3}})^5 - \cdots] =$ by equation (18) $= \sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}} = \mathbf{\sqrt{3}\frac{\pi}{6}}$.

36 $y'(0) = [y(0)]^2 = \mathbf{b^2}$; $y''(0) = 2y(0)y'(0) = \mathbf{2b^3}$; $y'''(0) =$ (from second derivative of $y' = y^2$) $= 2y(0)y''(0) + 2y'(0)y'(0) = \mathbf{6b^4}$. Then $y(x) = b + b^2x + 2b^3(\frac{x^2}{2}) + 6b^4(\frac{x^3}{6}) + \cdots = b(1 + bx + b^2x^2 + b^3x^3 + \cdots)$ which is $\mathbf{y(x) = \frac{b}{1-bx}}$.

38 The mean value is $\mu = \frac{1}{4} + 2(\frac{3}{4})(\frac{1}{4}) + 3(\frac{3}{4})^2(\frac{1}{4}) + \cdots =$ (by equation (5) with $x = \frac{3}{4}$) $= \frac{1}{4}\frac{1}{(1-\frac{3}{4})^2} = \mathbf{4}$. Why should you have to wait for the fourth deal to get the best hand??

40 Note: The equations referred to should be (10) and (13). Choose $x = \frac{2}{3}$ in (10a): $\frac{2}{3} + \frac{1}{2}(\frac{2}{3})^2 + \cdots = -\ln\frac{1}{3}$. In equation (13) choose $x = \frac{1}{2}$ so that $\frac{1+x}{1-x} = 3$. Then $2(\frac{1}{2} + \frac{1}{3}(\frac{1}{2})^3 + \frac{1}{5}(\frac{1}{2})^5 + \cdots) = \ln 3$. This converges faster because of the factor $(\frac{1}{2})^2$ between successive terms, compared to $\frac{2}{3}$ in the first series. **The series are equal** because $-\ln\frac{1}{3} = \ln 3$.

42 Equation (18) gives $\tan^{-1}\frac{1}{10} \approx \frac{1}{10} - \frac{1}{300} + \frac{1}{50000} \approx .10000 - .00333 + .00002 \approx \mathbf{.09669}$ (which is .0967 to four decimal places).

44 The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ adds to .693. The fact that the first digit is 6 is settled when the sum stays below .7, at about the term $+\frac{1}{71}$ (other answers are equally acceptable!). The **50th** power of $\frac{1}{4}$ equals the 100th power of $\frac{1}{2}$. Also $\frac{1}{a^n} = \frac{1}{2^{100}}$ when $a^n = 2^{100}$ or $n\ln a = 100\ln 2$ or $\mathbf{n = 100\frac{\ln 2}{\ln a}}$.

46 Equation (20) is $\pi = 4(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}) \approx 4(\frac{1}{2} - \frac{1}{3}(\frac{1}{2})^3 + \frac{1}{5}(\frac{1}{2})^5 - \frac{1}{7}(\frac{1}{2})^7 + \frac{1}{9}(\frac{1}{2})^9 - \frac{1}{11}(\frac{1}{2})^{11} + \frac{1}{3} - \frac{1}{3}(\frac{1}{3})^3 + \frac{1}{5}(\frac{1}{3})^5 - \frac{1}{7}(\frac{1}{3})^7) = 2 - .16667 + .02500 - .00446 + .00087 - .00018 + 1.33333 - .04938 + .00329 - .00026 = 3.1415^+ = \mathbf{3.142}$. Note: $\frac{1}{13}(\frac{1}{2})^{13}$ and $\frac{1}{9}(\frac{1}{3})^9$ will increase the total toward 3.1416.

48 $\sum \frac{e^{in}}{n} =$ (take $x = e^i$ in equation (10)) $= \mathbf{-\ln(1 - e^i)}$.

10.2 Convergence Tests: Positive Series (page 380)

The convergence of $a_1 + a_2 + \cdots$ is decided by the partial sums $s_n = \mathbf{a_1} + \cdots + \mathbf{a_n}$. If the s_n approach s, then $\sum a_n = \mathbf{s}$. For the **geometric** series $1 + x + \cdots$ the partial sums are $s_n = \mathbf{(1 - x^n)/(1 - x)}$. In that case $s_n \to 1/(1-x)$ if and only if $\mathbf{|x| < 1}$. In all cases the limit $s_n \to s$ requires that $a_n \to \mathbf{0}$. But the harmonic series $a_n = 1/n$ shows that we can have $a_n \to 0$ and still the series **diverges**.

The comparison test says that if $0 \le a_n \le b_n$ then $\sum \mathbf{a_n}$ **converges if** $\sum \mathbf{b_n}$ **converges**. In case a decreasing $y(x)$ agrees with a_n at $x = n$, we can apply the **integral** test. The sum $\sum a_n$ converges if and only if $\mathbf{\int_1^\infty y(x)dx}$ **converges**. By this test the p-series $\sum 1/n^p$ converges if and only if p is **greater than 1**. For the harmonic series $(p = 1)$, $s_n = 1 + \cdots + 1/n$ is near the integral $f(n) = \mathbf{\ln n}$.

The **ratio** test applies when $a_{n+1}/a_n \to L$. There is convergence if $\mathbf{|L| < 1}$, divergence if $\mathbf{|L| > 1}$, and no decision if $\mathbf{L = 1}$ **or** $\mathbf{-1}$. The same is true for the **root** test, when $(a_n)^{1/n} \to L$. For a geometric-p-series combination $a_n = x^n/n^p$, the ratio a_{n+1}/a_n equals $\mathbf{x(n+1)^p/n^p}$. Its limit is $L = \mathbf{x}$ so there is convergence if $|\mathbf{x}| < 1$. For the exponential $e^x = \sum x^n/n!$ the limiting ratio a_{n+1}/a_n or $x/(n+1)$ is $L = 0$. This series always converges because $n!$ grows faster than any x^n or n^p.

There is no sharp line between **convergence** and **divergence**. But if $\sum b_n$ converges and a_n/b_n approaches L, it follows from the **limit comparison** test that $\sum a_n$ also converges.

1 $\frac{1}{2} + \frac{1}{4} + \cdots$ is smaller than $1 + \frac{1}{3} + \cdots$
3 $a_n = s_n - s_{n-1} = \frac{1}{n^2-n}, s = 1; a_n = 4, s = \infty; a_n = \ln\frac{2n}{n+1} - \ln\frac{2(n-1)}{n} = \ln\frac{n^2}{n-1}, s = \ln 2$
5 No decision on $\sum b_n$ **7** Diverges: $\frac{1}{100}(1 + \frac{1}{2} + \cdots)$ **9** $\sum \frac{1}{100+n^2}$ converges: $\sum \frac{1}{n^2}$ is larger
11 Converges: $\sum \frac{1}{n^2}$ is larger **13** Diverges: $\sum \frac{1}{2n}$ is smaller **15** Diverges: $\sum \frac{1}{2n}$ is smaller
17 Converges: $\sum \frac{2}{2^n}$ is larger **19** Converges: $\sum \frac{3}{3^n}$ is larger **21** $L = 0$ **23** $L = 0$ **25** $L = \frac{1}{2}$
27 root $(\frac{n-1}{n})^n \to L = \frac{1}{e}$ **29** $s = 1$ (only survivor) **31** If y decreases, $\sum_2^n y(i) \le \int_1^n y(x)dx \le \sum_1^{n-1} y(i)$
33 $\sum_1^\infty e^{-n} \le \int^{\infty_0} e^{-x}dx = 1; \frac{1}{e} + \frac{1}{e^2} + \cdots = \frac{1}{e-1}$ **35** Converges faster than $\int \frac{dx}{x^2+1}$
37 Diverges because $\int_0^\infty \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln(x^2+1)]_0^\infty = \infty$ **39** Diverges because $\int_1^\infty x^{e-\pi}dx = \frac{x^{e-\pi+1}}{e-\pi+1}]_0^\infty = \infty$
41 Converges (geometric) because $\int_1^\infty (\frac{e}{\pi})^x dx < \infty$ **43** (b) $\int_n^{n+1} \frac{dx}{x} >$ (base 1) (height $\frac{1}{n+1}$)
45 After adding we have $1 + \frac{1}{2} + \cdots + \frac{1}{2n}$ (close to $\ln 2n$); thus originally close to $\ln 2n - \ln n = \ln\frac{2n}{n} = \ln 2$
47 $\int_{100}^{1000} \frac{dx}{x^2} = \frac{1}{100} - \frac{1}{1000} = .009$ **49** Comparison test: $\sin a_n < a_n$; if $a_n = \pi n$ then $\sin a_n = 0$ but $\sum a_n = \infty$
51 $a_n = n^{-5/2}$ **53** $a_n = \frac{2^n}{n^n}$ **55** Ratios are $1, \frac{1}{2}, 1, \frac{1}{2}, \cdots$ (no limit L); $(\frac{1}{2^k})^{1/2k} = \frac{1}{\sqrt{2}}$; yes
57 Root test $\frac{1}{\ln n} \to L = 0$ **59** Root test $L = \frac{1}{10}$ **61** Terms don't approach zero: Diverge
63 Diverge (compare $\sum \frac{1}{n}$) **65** Root test $L = \frac{3}{4}$ **67** Beyond some point $\frac{a_n}{b_n} < 1$ or $a_n < b_n$

2 The series $\frac{a}{10} + \frac{b}{100} + \frac{c}{1000} + \cdots$ converges because it is below the comparison series $\mathbf{\frac{9}{10} + \frac{9}{100} + \frac{9}{1000}} \cdots = .999\cdots = 1$.
4 (a) $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^{n-1}} = \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}}$. (b) $\ln\frac{1}{2} + \ln\frac{2}{3} + \cdots + \ln\frac{n}{n+1} = \ln(\frac{1}{2})(\frac{2}{3})\cdots(\frac{n}{n+1}) = \ln\mathbf{\frac{1}{n+1}}$
(c) $1 + 2 + \cdots + n = \mathbf{\frac{1}{2}n(n+1)}$
6 (a) If $b_n + c_n < a_n$ (all positive) and $\sum a_n$ converges then by comparison $\sum \mathbf{b_n}$ **and** $\sum \mathbf{c_n}$ **converge.**
(b) If $a_n < b_n + c_n$ (all positive) and $\sum a_n$ diverges then $\sum \mathbf{b_n}$ **or** $\sum \mathbf{c_n}$ **(or both) must diverge.**
8 $\frac{1}{100} + \frac{1}{105} + \frac{1}{110} + \cdots$ **diverges** by comparison with $\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \cdots$ in Problem 7.
10 $\frac{1}{101} + \frac{2}{108} + \frac{3}{127}$ $(+ \frac{n}{100+n^3})$ **converges** by comparison with $\sum \frac{1}{n^2}$. (Just drop the 100.)
12 $\sum \frac{1}{\sqrt{n^2+10}}$ **diverges** by comparison with the smaller series $\sum \frac{1}{n+10}$ which diverges. Check that $\sqrt{n^2+10}$ is less than $n + 10$.
14 $\sum \frac{\sqrt{n}}{n^2+4}$ **converges** because it is less than $\sum \frac{\sqrt{n}}{n^2} = \sum \frac{1}{n^{3/2}}$ which converges.
16 $\sum \frac{1}{n^2}\cos(\frac{1}{n})$ **converges** because it is below $\sum \frac{1}{n^2}$.
18 $\sum \sin^2(\frac{1}{n})$ **converges** because $\sin\frac{1}{n} < \frac{1}{n}$ gives $\sin^2(\frac{1}{n}) < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges.
20 $\sum \frac{1}{e^n - n^e}$ **converges** because $\frac{n^e}{e^n} \to 0$, which means that eventually $n^e < \frac{1}{2}e^n$ and the series compares with $\sum \frac{1}{\frac{1}{2}e^n}$ which converges.
22 The limit of $\frac{1}{(n+1)^2}/\frac{1}{n^2} = \frac{n^2}{(n+1)^2} = \mathbf{L = 1}$. So the ratio test (and root test) give no decision.
24 The terms are $(\frac{n-1}{n})^n = (1 - \frac{1}{n})^n \to e^{-1}$ so the ratio approaches $\mathbf{L = \frac{e^{-1}}{e^{-1}} = 1}$. (Divergent series because its terms don't approach zero.)
26 The ratios $\frac{(n+1)!}{e^{(n+1)^2}}/\frac{n!}{e^{n^2}} = \frac{n+1}{e^{2n+1}}$ approach $\mathbf{L = 0}$.
28 The ratios $\frac{(n+1)!}{(n+1)^{n+1}}/\frac{n!}{n^n} = \frac{(n+1)}{(n+1)}(\frac{n}{n+1})^n = (\frac{n+1}{n})^{-n}$ approach $\mathbf{L = e^{-1}}$.
30 (a) Put $(\frac{1}{1} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{5}) + \cdots = 1$ together with $(\frac{1}{2} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{6}) + \cdots = \frac{1}{2}$ to obtain $\mathbf{s = 1 + \frac{1}{2} = \frac{3}{2}}$.
(b) $\ln\frac{1}{2} + \ln\frac{2}{3} + \cdots + \ln\frac{n}{n+1} = \ln(\frac{1}{2})(\frac{2}{3})\cdots(\frac{n}{n+1}) = \ln\frac{1}{n+1}$ approaches $\ln 0 = -\infty$ (no sum s).

32 $1+\frac{1}{3}+\cdots+\frac{1}{2n-1}$ is the area of rectangles outside $\int_1^{n+1}\frac{dx}{2x-1}=[\frac{1}{2}\ln(2x-1)]_1^{n+1}=\frac{1}{2}\ln(2n+1)$. The rectangular area is less than $1+\int_1^n\frac{dx}{2x-1}=1+\frac{1}{2}\ln(2n-1)$. Similarly $1+\frac{1}{8}+\cdots+\frac{1}{n^3}$ is larger than $\int_1^{n+1}\frac{dx}{x^3}=[-\frac{1}{2x^2}]_1^{n+1}=\frac{1}{2}-\frac{1}{2(n+1)^2}$. The sum is smaller than $1+\int_1^n\frac{dx}{x^3}=\frac{3}{2}-\frac{1}{2n^2}$.

34 The sum $\frac{1}{e}+\frac{2}{e^2}+\cdots$ is less than $\frac{1}{e}+\int_1^\infty xe^{-x}dx=\frac{1}{e}+[-xe^{-x}-e^{-x}]_1^\infty=\frac{3}{e}$. (Note that xe^{-x} decreases for $x>1$, so $\frac{2}{e^2}$ is less than the integral from 1 to 2.) The exact sum is in equation (6) of Section 10.1: $x+2x^2+\cdots=\frac{x}{(1-x)^2}=\frac{\frac{1}{e}}{(1-\frac{1}{e})^2}=\frac{e}{(e-1)^2}$.

36 $\sum\frac{1}{3n+5}$ diverges by comparison with $\int_1^\infty\frac{dx}{3x+5}=\frac{1}{3}\ln(3x+5)]_1^\infty=\infty$.

38 $\sum\frac{\ln n}{n}$ diverges by comparison with $\int_1^\infty\frac{\ln x}{x}dx=[\frac{1}{2}(\ln x)^2]_1^\infty=\infty$. ($\frac{\ln x}{x}$ decreases for $x>e$ and these later terms decide divergence. Another comparison is with $\sum\frac{1}{n}$.)

40 $\sum_2^\infty\frac{1}{n(\ln n)(\ln\ln n)}$ diverges by comparison with $\int\frac{dx}{x(\ln x)(\ln\ln x)}=\ln(\ln\ln x)]_2^\infty=\infty$.

42 $\sum_1^\infty\frac{n}{e^{n^2}}$ converges by comparison with $\int_1^\infty\frac{x\,dx}{e^{x^2}}=-\frac{1}{2}e^{-x^2}]_1^\infty=\frac{1}{2e}$.

44 The partial sum $1+\frac{1}{2}+\cdots+\frac{1}{n}$ is near $.577+\ln n$. This exceeds 7 when $\ln n>6.423$ or $n>e^{6.423}>615$. The sum exceeds 10 when $\ln n>9.423$ or $n>e^{9.423}>12369$.

46 The first term is $\frac{1}{2\ln 2}$. After that $\frac{1}{n\ln n}<\int_{n-1}^n\frac{dx}{x\ln x}$. The sum from 3 to n is below $\int_2^n\frac{dx}{x\ln x}=\ln(\ln n)-\ln(\ln 2)$. By page 377 the computer has not reached, $n=3.2\cdot 10^{19}$ in a million years. So the sum has not reached $\frac{1}{2\ln 2}+\ln(\ln 3.2\cdot 10^{19})-\ln(\ln 2)<5$.

48 If $\sum a_n$ converges then all $a_n<1$ beyond some point $n=N$. Therefore $a_n^2<a_n$ beyond this point and $\sum a_n^2$ converges by comparison with $\sum a_n$.

50 The limit comparison test says that $\sum\frac{1}{p_n}$ diverges if $\sum\frac{1}{n\ln n}$ diverges. The integral test says that $\sum\frac{1}{n\ln n}$ diverges because $\int^\infty\frac{dx}{x\ln x}=\ln(\ln x)]^\infty=\infty$.

52 $a_n=\sqrt{n}(\frac{1}{2})^n$ has $\frac{a_n}{b_n}\to 0,\frac{a_n}{c_n}\to\infty$.

54 $a_n=\frac{1}{2^n}$ has $\frac{a_n}{b_n}=\frac{n^e}{2^n}\to 0$ but $\frac{a_n}{c_n}=(\frac{e}{2})^n\to\infty$.

56 Suppose $\frac{a_{n+1}}{a_n}$ is between $L-\epsilon$ and $L+\epsilon$ for $n>N$. This is true for all n if we change the first terms of the sequence to $a_n=a_NL^{n-N}(n=0,1,\cdots,N)$. Then the products $(\frac{a_1}{a_0})(\frac{a_2}{a_1})\cdots(\frac{a_n}{a_{n-1}})=\frac{a_n}{a_0}$ are between $(L-\epsilon)^n$ and $(L+\epsilon)^n$. Take the nth root: $a_n^{1/n}$ is between $a_0^{1/n}(L-\epsilon)$ and $a_0^{1/n}(L+\epsilon)$. For small ϵ and large n this nth root is arbitrarily close to L.

58 $\sum\frac{1}{n^{\ln n}}$ converges by comparison with $\sum\frac{2}{n^2}$ (note $\ln n>2$ beyond the 8th term). (Also $\sum\frac{1}{(\ln n)^{\ln n}}$ converges!)

60 $\ln 10^n=n\ln 10$ so the sum is $\frac{1}{\ln 10}\sum\frac{1}{n}=\infty$ (harmonic series diverges)

62 $n^{-1/n}$ approaches 1 so the series cannot converge

64 $\sum\frac{\ln n}{n^p}$ diverges by comparison with $\sum\frac{1}{n^p}$ if $p\le 1$. For $p>1$ the terms $\frac{\ln n}{n^p}$ are eventually smaller than $\frac{1}{n^P}$ with $1<P<p$. So $\sum\frac{\ln n}{n?}$ converges if $p>1$.

66 $\sum\frac{n^p}{(n!)^q}$ converges if $q>0$ by the ratio test: $\frac{(n+1)^p}{((n+1)!)^q}/\frac{n^p}{(n!)^q}=(\frac{n+1}{n})^p\frac{1}{(n+1)^q}\to L=0$. **68** No.

10.3 Convergence Tests: All Series (page 384)

The series $\sum a_n$ is absolutely convergent if the series $\sum|\mathbf{a_n}|$ is convergent. Then the original series $\sum a_n$ is also **convergent**. But the series $\sum a_n$ can converge without converging absolutely. That is called **conditional** convergence, and the series $\mathbf{1-\frac{1}{2}+\frac{1}{3}}-\cdots$ is an example.

For alternating series, the sign of each a_{n+1} is **opposite** to the sign of a_n. With the extra conditions that $|\mathbf{a_{n+1}}|\le|\mathbf{a_n}|$ and $\mathbf{a_n\to 0}$, the series converges (at least conditionally). The partial sums $s_1,s_3,\cdots$ are

decreasing and the partial sums $s_2, s_4, \cdots$ are **increasing.** The difference between s_n and s_{n-1} is $\mathbf{a_n}$. Therefore the two series converge to the same number s. An alternating series that converges absolutely [conditionally] (not at all) is $\sum(-1)^{n+1}/n^2[\sum(-1)^{n+1}/n](\sum(-1)^{n+1})$. With absolute [conditional] convergence a reordering cannot [can] change the sum.

1 Conditionally not absolutely **3** Absolutely **5** Conditionally not absolutely **7** No convergence
9 Absolutely **11** No convergence **13** By comparison with $\sum|a_n|$
15 Even sums $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots$ diverge; a_n's are not decreasing **17** (b) If $a_n > 0$ then s_n is too large so $s - s_n < 0$
19 $s = 1 - \frac{1}{e}$; below by less than $\frac{1}{5!}$
21 Subtract $2(\frac{1}{2^2}+\frac{1}{4^2}+\cdots) = \frac{2}{4}(\frac{1}{1^2}+\frac{1}{2^2}+\cdots) = \frac{\pi^2}{12}$ from positive series to get alternating series
23 Text proves: If $\sum|a_n|$ converges so does $\sum a_n$
25 New series $= (\frac{1}{2}) - \frac{1}{4} + (\frac{1}{6}) - \frac{1}{8}\cdots = \frac{1}{2}(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots)$ **27** $\frac{3}{2}\ln 2$: add $\ln 2$ series to $\frac{1}{2}$ ($\ln 2$ series)
29 Terms alternate and decrease to zero; partial sums are $1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n \to \gamma$
31 .5403? **33** Hint + comparison test **35** Partial sums $a_n - a_0$; sum $-a_0$ if $a_n \to 0$
37 $\frac{1}{1-\frac{1}{2}}\frac{1}{1-\frac{1}{3}} = 3$ but product is not $1+\frac{2}{3}+\cdots$
39 Write x to base 2, as in 1.0010 which keeps $1+\frac{1}{8}$ and deletes $\frac{1}{2},\frac{1}{4},\cdots$
41 $\frac{1}{9}+\frac{1}{27}+\cdots$ adds to $\frac{1/9}{1-1/3} = \frac{1}{6}$ and can't cancel $\frac{1}{3}$
43 $\frac{\sin 1}{1-\cos 1} = \cot\frac{1}{2}$ (trig identity) $= \tan(\frac{\pi}{2}-\frac{1}{2})$; $s = \sum\frac{e^{in}}{n} = -\log(1-e^i)$ by 10a in Section 10.1; take imaginary part

2 $\sum\frac{(-1)^{n-1}}{\sqrt{n+3}}$: **converges conditionally** (passes alternating series test) but not absolutely: $\sum\frac{1}{\sqrt{n}}$ diverges
4 $\sum\frac{3^n}{n!}$ converges (ratio test: $\frac{a_{n+1}}{a_n} = \frac{3}{n+1} \to 0$) so there is **absolute convergence.**
6 $\sum(-1)^{n+1}\sin^2 n$ **diverges** (terms don't approach zero)
8 $\sum(-1)^{n+1}\frac{\sin^2 n}{n}$: no absolute convergence because $\sin^2 n > \frac{1}{2}$ half of the time and $\sum\frac{1}{2n}$ diverges. The terms alternate in sign but do not decrease steadily; still I believe there is conditional convergence.
10 $\sum(-1)^{n+1}2^{1/n}$ **diverges** (terms don't approach zero)
12 $n^{1/n}$ decreases steadily to 1 so the alternating test is passed: $\sum(-1)^{n+1}(1-n^{1/n})$ converges **conditionally.** But $n^{1/n} > e^{1/n} > (1+\frac{1}{n})$ so that $|1-n^{1/n}|$ exceeds $\frac{1}{n}$ and there is no absolute convergence.
14 **Yes**, the sum $\sum(-\frac{1}{n^2})$ converges absolutely.
16 The terms alternate in sign but do not decrease to zero. The positive terms $\frac{2}{3},\frac{4}{7},\frac{6}{11},\cdots$ approach $\frac{1}{2}$ and so does the sequence $\frac{3}{5},\frac{5}{9},\frac{7}{13},\cdots$
18 The term after $s_5 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5} = \frac{47}{60}$ is $a_6 = -\frac{1}{6}$. Then later terms bring the sum upward. So the sum $s = \ln 2$ is between $\frac{47}{60}-\frac{1}{6} = \frac{37}{60}$ and $\frac{47}{60}$.
20 The difference between s and s_{100} is less than $\mathbf{\frac{1}{101^2}}$, the next term in the series (because after that term comes $-\frac{1}{102^2}$ and the sums stay between s_{100} and s_{101}).
22 The error $\frac{1}{101^2}-\frac{1}{102^2}+\cdots$ in the **alternating** series is smaller than the error in the positive series.
24 The series $a_1+a_2-a_3+a_4+a_5-a_6+\cdots$ is sure to converge (*conditionally*) if $0 \le a_{3n+3} < a_{3n+1}+a_{3n+2} < a_{3n}$ for every n. Then it passes the alternating series test when each pair of positive terms is combined. (The series could converge without passing this particular test.)
26 The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots = \ln 2$ is in Section 10.1. Take half of every term and also insert zeros: $0+\frac{1}{2}-0-\frac{1}{4}+0+\frac{1}{6}-\cdots = \frac{1}{2}\ln 2$. Add the two series term by term: $1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\cdots = \frac{3}{2}\ln 2$, as the problem requires. This is allowed because the partial sums s_n' and s_n'' of the first two series add to the partial sums s_n of the third series. Notice something strange: The third series can also be produced from the first series only, by rearranging (two positive terms between negative terms). With

conditional convergence any sum is possible.

28 Shorter answer than expected: $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}$ comes from rearranging $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}$. Continue this way, **six terms at a time.** The partial sums $s_6, s_{12}, \cdots$ are not changed and still approach ln 2. The partial sums in between also approach ln 2 because the six terms in each group approach zero.

30 Apply the alternating test. The terms $\int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x} dx$ are $+,-,+,-,\cdots$ (because $\sin x$ alternates). The terms are decreasing and approach zero (because of $\frac{1}{x}$). Why is the sum $\frac{\pi}{2}$?

32 We know that $\sin \pi = 0$, or $\pi - \frac{\pi^3}{6} + \frac{\pi^5}{120} - \cdots = 0$. If we stop just before the term $\pm \frac{\pi^{2n+1}}{(2n+1)!}$, the error is less than 10^{-8} (or $\frac{1}{2}10^{-8}$ to be safe) if $\frac{\pi^{2n+1}}{(2n+1)!} < \frac{1}{2}10^{-8}$ which is true for $n = 10$.

34 The series can start at $n = 1$ or $n = 0$ (we choose $n = 0$ to have geometric series): $\sum a_n^2 = 1+\frac{1}{4}+\frac{1}{16}+\cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$ and $\sum b_n^2 = 1+\frac{1}{9}+\frac{1}{81}+\cdots = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$ and $\sum a_n b_n = 1+\frac{1}{6}+\frac{1}{36}+\cdots = \frac{1}{1-\frac{1}{6}} = \frac{6}{5}$. Check the Schwarz inequality: $(\frac{6}{5})^2 < (\frac{4}{3})(\frac{9}{8})$ or $6 \cdot 6 \cdot 3 \cdot 8 < 5 \cdot 5 \cdot 4 \cdot 9$ or $864 < 1125$.

36 If $\sum a_n$ is conditionally but not absolutely convergent, take positive terms until the sum exceeds 10. Then take one negative term. Then positive terms until the sum exceeds 20. Then one negative term, and so on. The partial sums approach $+\infty$ (because the single negative terms go to zero, otherwise no conditional convergence in the first place).

38 (a) **False** ($1-1+1-1+\cdots$ does not converge) (b) **False** (same example) (c) **True** (d) **True** ($a_1 + \cdots + a_N$ added to $b_1 + \cdots + b_N$ equals $(a_1+b_1)+\cdots+(a_N+b_N)$; let $N \to \infty$; then $\sum a_n + \sum b_n = \sum(a_n+b_n)$).

40 For $s = -1$ choose all minus signs: $-\frac{1}{2}-\frac{1}{4}-\cdots = -1$. For $s = 0$ choose one plus sign and then all minus: $\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\cdots = 0$. For $s = \frac{1}{3}$ choose alternating signs: $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\cdots = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$.

42 The smallest positive number must include $+\frac{1}{3}$; then choose all minus signs: $\frac{1}{3}-\frac{1}{9}-\frac{1}{27}-\cdots = \frac{1}{3}-\frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{1}{3}-\frac{1}{6} = \frac{1}{6}$. (This is for the Cantor set centered at zero. Add $\frac{1}{2}$ to obtain the number $\frac{2}{3}$ in the usual Cantor set between 0 and 1.) With **alternating signs** the sum $\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\cdots = \frac{\frac{1}{3}}{1+\frac{1}{3}} = \frac{1}{4}$.

44 If $\sum a_n$ converges then its terms approach zero: in particular $|a_n| \le C$ for some number C. Then $\sum a_n x^n$ converges absolutely by comparison with $\sum C|x|^n = \frac{C}{1-|x|}$.

10.4 The Taylor Series for e^x, sin x and cos x (page 390)

The **Taylor** series is chosen to match $f(x)$ and all its **derivatives** at the basepoint. Around $x = 0$ the series begins with $f(0) + \mathbf{f'(0)}x + \frac{1}{2}\mathbf{f''(0)}x^2$. The coefficient of x^n is $\mathbf{f^{(n)}(0)/n!}$. For $f(x) = e^x$ this series is $\sum \mathbf{x^n/n!}$. For $f(x) = \cos x$ the series is $\mathbf{1 - x^2/2! + x^4/4! - \cdots}$. For $f(x) = \sin x$ the series is $\mathbf{x - x^3/3! + \cdots}$. If the signs were positive in those series, the functions would be $\cosh x$ and **sinh x**. Addition gives $\cosh x + \sinh x = \mathbf{e^x}$.

In the Taylor series for $f(x)$ around $x = a$, the coefficient of $(x-a)^n$ is $b_n = \mathbf{f^{(n)}(a)/n!}$. Then $b_n(x-a)^n$ has the same **derivatives** as f at the basepoint. In the example $f(x) = x^2$, the Taylor coefficients are $b_0 = \mathbf{a^2}, b_1 = \mathbf{2a}, b_2 = \mathbf{1}$. The series $b_0 + b_1(x-a) + b_2(x-a)^2$ agrees with the original $\mathbf{x^2}$. The series for e^x around $x = a$ has $b_n = \mathbf{e^a/n!}$. Then the Taylor series reproduces the identity $e^x = (\mathbf{e^a})(\mathbf{e^{x-a}})$.

We define e^x, $\sin x$, $\cos x$, and also $e^{i\theta}$ by their series. The derivative $d/dx(1 + x + \frac{1}{2}x^2 + \cdots) = 1 + x + \cdots$ translates to $\mathbf{d/dx(e^x) = e^x}$. The derivative of $1 - \frac{1}{2}x^2 + \cdots$ is $\mathbf{-x + x^3/3! - \cdots}$. Using $i^2 = -1$ the series $1 + i\theta + \frac{1}{2}(i\theta)^2 + \cdots$ splits into $e^{i\theta} = \mathbf{\cos\theta + i\sin\theta}$. Its square gives $e^{2i\theta} = \mathbf{\cos 2\theta + i \sin 2\theta}$. Its reciprocal is $e^{-i\theta} = \mathbf{\cos\theta - i\sin\theta}$. Multiplying by r gives $re^{i\theta} = \mathbf{r\cos\theta + ir\sin\theta}$, which connects the polar and rectangular forms of a complex number. The logarithm of $e^{i\theta}$ is $i\theta$.

1 $1+2x+\frac{(2x)^2}{2!}+\cdots$; derivatives 2^n; $1+2+\frac{2^2}{2!}+\cdots$ **3** Derivatives i^n; $1+ix+\cdots$
5 Derivatives $2^n n!$; $1+2x+4x^2+\cdots$ **7** Derivatives $-(n-1)!$; $-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$
9 $y=2-e^x=1-x-\frac{x^2}{2!}-\cdots$ **11** $y=x-\frac{x^3}{6}+\cdots=\sin x$ **13** $y=xe^x=x+x^2+\frac{x^3}{2!}+\cdots$
15 $1+2x+x^2$; $4+4(x-1)+(x-1)^2$ **17** $-(x-1)^5$ **19** $1-(x-1)+(x-1)^2-\cdots$
21 $(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots=\ln(1+(x-1))$ **23** $e^{-1}e^{1-x}=e^{-1}(1-(x-1)+\frac{(x-1)^2}{2!}-\cdots)$
25 $x+2x^2+2x^3$ **27** $\frac{1}{2}-\frac{x^2}{24}+\frac{x^4}{720}$ **29** $x-\frac{x^3}{18}+\frac{x^5}{600}$ **31** $1+x^2+\frac{x^4}{2}$ **33** $1+x-\frac{x^3}{2}$
35 ∞ slope; $1+\frac{1}{2}(x-1)$ **37** $x-\frac{x^3}{3}+\frac{x^5}{5}$ **39** $x+\frac{x^3}{3}+\frac{2x^5}{15}$ **41** $1+x+\frac{x^2}{2}$ **43** $1+0x-x^2$
45 $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$ **47** 99th powers $-1, -i, e^{3\pi i/4}, -i$
49 $e^{-i\pi/3}$ and -1; sum zero, product -1 **53** $i\frac{\pi}{2}, i\frac{\pi}{2}+2\pi i$ **55** $2e^x$

2 $\sin 2x=2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots$ so that $(\sin 2x)'''=-2^3+\frac{2^5x^2}{2!}-\cdots=-8$ at $x=0$. This agrees with the chain rule for $(\sin 2x)'''$. Also $\sin 2\pi=2\pi-\frac{(2\pi)^3}{3!}+\frac{(2\pi)^5}{5!}-\cdots=0$.

4 $f=\frac{1}{1+x}, f'=\frac{-1}{(1+x)^2}, f''=\frac{2!}{(1+x)^3}, f'''=\frac{-3!}{(1+x)^4},\cdots$. Set $x=0: f=1, f'=-1, f''=2!, f'''=-3!,\cdots$. The Taylor series is $\frac{1}{1+x}=1-x+\frac{2!}{2!}x^2-\frac{3!}{3!}x^3+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n$.

6 $f=\cosh x, f'=\sinh x, f''=\cosh x,\cdots$. Evaluate at $x=0: f=1, f'=0, f''=1,\cdots$. The Taylor series is $\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$.

8 $f=\ln(1+x), f'=\frac{1}{1+x}, f''=\frac{-1}{(1+x)^2},\cdots$ (one step behind Problem 4). Evaluate at $x=0: f=0, f'=1, f''=-1, f'''=2!,\cdots$. The Taylor series is $\ln(1+x)=x-\frac{x^2}{2!}+\frac{2!x^3}{3!}-\cdots=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$.

10 $y'=cy+s, y''=cy', y'''=cy'',\cdots$. With $y_0=0$ this gives $y_0'=s, y_0''=cs, y_0'''=c^2s,\cdots$. The Taylor series is $y(x)=sx+cs\frac{x^2}{2!}+c^2s\frac{x^3}{3!}+\cdots=\frac{s}{c}[cx+c^2\frac{x^2}{2!}+c^3\frac{x^3}{3!}+\cdots]=\frac{s}{c}(e^{cx}-1)$.

12 $y'=y$ yields $y''=y'=y$ and $y'''=y\cdots$. Then y and all its derivatives equal 1 at $x=3$. The Taylor series is $y(x)=1+(x-3)+\frac{1}{2!}(x-3)^2+\cdots=e^{x-3}$.

14 At $x=0$ the equation gives $y''=y=1$, $y'''=y'=0$ and $y''''=y''=y=1$ (even derivatives equal 1, odd derivatives equal 0). The Taylor series is $y(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots=\frac{1}{2}(e^x+e^{-x})=\cosh x$.

16 x^3 and its derivatives at $x=a$ are $a^3, 3a^2, 6a, 6, 0,\cdots$. The Taylor series is $a^3+3a^2(x-a)+\frac{6a}{2}(x-a)^2+\frac{6}{6}(x-a)^3$ which agrees with x^3.

18 At $x=2\pi$ the cosine and its derivatives are $1,0,-1,0,1,\cdots$. The Taylor series is $\cos x=1-\frac{(x-2\pi)^2}{2!}+\frac{(x-2\pi)^4}{4!}-\cdots$. At $x=0$ the function $\cos(x-2\pi)$ and its derivatives again equal $1,0,-1,0,1,\cdots$. Now the Taylor series is $\cos(x-2\pi)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$.

20 $\frac{1}{2-x}$ has derivatives $\frac{1}{(2-x)^2},\frac{2}{(2-x)^3},\frac{6}{(2-x)^4},\cdots$. At $x=1$ those equal $1,1,2,6,\cdots$ and the series is $\frac{1}{2-x}=1+(x-1)+(x-1)^2+(x-1)^3+\cdots$.

22 At $x=1$ the function x^4 and its derivatives equal 1,4,12,24,0,0,$\cdots$. The Taylor series has five nonzero terms: $x^4=1+4(x-1)+\frac{12}{2}(x-1)^2+\frac{24}{6}(x-1)^3+\frac{24}{24}(x-1)^4$.

24 The function e^{2x} has derivatives $2e^{2x}, 4e^{2x}, 8e^{2x},\cdots$. Evaluating at $x=1$ gives $e^2, 2e^2, 4e^2, 8e^2,\cdots$. The Taylor series is $e^{2x}=e^2+2e^2(x-1)+4e^2\frac{(x-1)^2}{2!}+8e^2\frac{(x-1)^3}{3!}+\cdots$ (which is e^2 times $e^{2(x-1)}$).

26 $\cos\sqrt{x}=1-\frac{(\sqrt{x})^2}{2!}+\frac{(\sqrt{x})^4}{4!}-\cdots=1-\frac{x}{2}+\frac{x^2}{24}-\cdots$. (Note that $\sin\sqrt{x}$ would not succeed; the terms $\sqrt{x}, (\sqrt{x})^3,\cdots$ are not acceptable in a Taylor series. The function has no derivative at $x=0$.)

28 $\frac{\sin x}{x}=\frac{x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots}{x}=1-\frac{x^2}{6}+\frac{x^4}{120}-\cdots$.

30 $\sin x^2=x^2-\frac{(x^2)^3}{6}+\frac{(x^2)^5}{120}-\cdots=x^2-\frac{x^6}{6}+\frac{x^{10}}{120}-\cdots$.

32 $b^x=e^{x\ln b}=1+x\ln b+\frac{1}{2}(x\ln b)^2+\cdots$.

34 $|x| = \begin{cases} -x & \text{for } x < 0 \\ +x & \text{for } x > 0 \end{cases}$ so that $\frac{d|x|}{dx} = \begin{cases} -1 & \text{for } x < 0 \\ +1 & \text{for } x > 0 \end{cases}$. When x is negative and n is not a whole number, x^n is a complex number. But still $\frac{dx^n}{dx} = n|x|^{n-1}e^{in\pi}(-1) = n|x|^{n-1}(e^{i\pi})^{n-1} = nx^{n-1}$.

36 $2^x = e^{x\ln 2} = 1 + x\ln 2 + \frac{1}{2!}(x\ln 2)^2 + \frac{1}{3!}(x\ln 2)^3 + \cdots$ (OK to compute derivatives).

38 Compute $\sin^{-1} x$ and its derivatives at $x = 0$: $\sin^{-1} x = 0$, $\frac{1}{\sqrt{1-x^2}} = 1$, $x(1-x^2)^{-3/2} = 0$, $(1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2} = 1, 9x(1-x^2)^{-5/2} - 15x^3(1-x^2)^{-7/2} = 0, 9(1-x^2)^{-5/2} + \cdots = 9$. The Taylor series for $\sin^{-1} x$ starts with $\mathbf{0 + x + 0 + \frac{1}{6}x^3 + 0 + \frac{9}{120}x^5}$.

40 Compute $\ln(\cos x)$ and its derivatives at $x = 0$: $\ln 1 = 0, -\frac{\sin x}{\cos x} = -\tan x = 0, -\sec^2 x = -1, -2\sec^2 x \tan x = 0, -2\sec^4 x - 4\sec^2 x \tan^2 x = -2$. The Taylor series for $\ln(\cos x)$ starts with $\mathbf{-\frac{1}{2}x^2 + 0 - \frac{2}{24}x^4}$.

42 Compute $\tanh^{-1} x$ (or $\frac{1}{2}\ln(\frac{1+x}{1-x})$: Section 6.7) and its derivatives at $x = 0$: $\tanh^{-1} 0 = 0$, $\frac{1}{1-x^2} = 1$, $2x(1-x^2)^{-2} = 0, 2(1-x^2)^{-2} + 4x^2(1-x^2)^{-3} = 2$. The series for $\tanh^{-1} x$ starts with $\mathbf{x + 0 + \frac{2}{6}x^3}$.

44 Compute $\sec^2 x$ and its derivatives at $x = 0$: $\sec^2 0 = 1, 2\sec^2 x \tan x = 0, 2\sec^4 x + 2\sec^2 x \tan^2 x = 2$. The Taylor series for $\sec^2 x$ starts with $1 + 0x + \frac{2}{2}x^2 = \mathbf{1 + x^2}$.

46 $(e^{i\theta})^2 = e^{2i\theta}$ equals $\cos 2\theta + i\sin 2\theta$, so **neither** of the proposed answers is correct.

48 (a) $e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \mathbf{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$ (b) $(e^{2\pi i/3})^3 = e^{6\pi i/3} = e^{2\pi i} = 1$ (c) $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = \frac{1}{4} - \frac{i\sqrt{3}}{2} - \frac{3}{4} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. Multiply by another $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})$ to get $\frac{1}{4} - i^2\frac{3}{4} = \mathbf{1}$.

50 $(2e^{i\pi/3})^2 = \mathbf{4e^{2\pi i/3}}$ and also $(1+\sqrt{3}i)(1+\sqrt{3}i) = 1 + 2\sqrt{3}i - 3 = \mathbf{-2 + 2\sqrt{3}i}$; $(4e^{i\pi/4})^2 = \mathbf{16e^{i\pi/2}}$ and also $(2\sqrt{2} + i2\sqrt{2})(2\sqrt{2} + i2\sqrt{2}) = 8 + 16i - 8 = \mathbf{16i}$.

52 Write $(e^{is})(e^{-it}) = e^{i(s-t)}$ in rectangular form: $(\cos s + i\sin s)(\cos t - i\sin t) = \cos(s-t) + i\sin(s-t)$. Collect real and imaginary parts: $\cos(s-t) = \mathbf{\cos s \cos t + \sin s \sin t}$ and $\sin(s-t) = \mathbf{\sin s \cos t - \cos s \sin t}$.

54 If $e = \frac{p}{q}$ then the number $N = p![\frac{1}{e} - (1 - \frac{1}{1!} + \frac{1}{2!} \cdots \pm \frac{1}{p!})]$ is an integer, because **all denominators go evenly into the p! term.** But in parentheses is an alternating and decreasing series approaching $e^{-1} = \frac{1}{e}$. The error is less than the last term $\frac{1}{p!}$ so $|N| < 1$. The only possible integer N is $N = 0$ which is not correct. The contradiction means that $e = \frac{p}{q}$ was not true: e is not a fraction.

10.5 Power Series (page 395)

If $|x| < |X|$ and $\sum a_n X^n$ converges, then the series $\sum a_n x^n$ also **converges.** There is convergence in a **symmetric** interval around the **origin.** For $\sum(2x)^n$ the convergence radius is $r = \mathbf{\frac{1}{2}}$. For $\sum x^n/n!$ the radius is $r = \infty$. For $\sum(x-3)^n$ there is convergence for $|x - 3| < \mathbf{1}$. Then x is between **2** and **4**.

Starting with $f(x)$, its Taylor series $\sum a_n x^n$ has $a_n = \mathbf{f^{(n)}(0)/n!}$. With basepoint a, the coefficient of $(x-a)^n$ is $\mathbf{f^{(n)}(a)/n!}$. The error after the x^n term is called the **remainder** $R_n(x)$. It is equal to $\mathbf{f^{(n+1)}(c)(x-a)^{n+1}/(n+1)!}$ where the unknown point c is between **a and x**. Thus the error is controlled by the $\mathbf{(n+1)}$**st** derivative.

The circle of convergence reaches out to the first point where $f(x)$ fails. For $f = 4/(2-x)$, that point is $x = \mathbf{2}$. Around the basepoint $a = 5$, the convergence radius would be $r = \mathbf{3}$. For $\sin x$ and $\cos x$ the radius is $r = \infty$.

The series for $\sqrt{1+x}$ is the **binomial** series with $p = \frac{1}{2}$. Its coefficients are $a_n = \mathbf{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots/n!}$. Its convergence radius is **1**. Its square is the very short series $1 + x$.

1 $1 + 4x + (4x)^2 + \cdots; r = \frac{1}{4}; x = \frac{1}{4}$ **3** $e(1 - x + \frac{x^2}{2!} - \cdots); r = \infty$

5 $\ln e + \ln(1 + \frac{x}{e}) = 1 + \frac{x}{e} - \frac{1}{2}(\frac{x}{e})^2 + \cdots; r = e; x = -e$

7 $|\frac{x-1}{2}| < 1$ or $(-1, 3)$; $\frac{2}{3-x}$ 9 $|x - a| < 1; -\ln(1 - (x - a))$

11 $1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$; add to 1 at $x = 0$ 13 $a_1, a_3, \cdots$ are all zero 15 $\frac{1-(1-\frac{1}{2}x^2\cdots)}{x^2} \to \frac{1}{2}$

17 $f^{(8)}(c) = \cos c < 1$; alternating terms might not decrease (as required)

19 $f = \frac{1}{1-x}, |R_n| \le \frac{x^{n+1}}{(1-c)^{n+2}}; R_n = \frac{x^{n+1}}{1-x}; (1-c)^4 = 1 - \frac{1}{2}$

21 $f^{(n+1)}(x) = \frac{n!}{(1-x)^{n+1}}, |R_n| \le \frac{x^{n+1}}{(1-c)^{n+1}}(\frac{1}{n+1}) \to 0$ when $x = \frac{1}{2}$ and $1 - c > \frac{1}{2}$

23 $R_2 = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2$ so $R_2 = R_2' = R_2'' = 0$ at $x = a, R_2''' = f'''$; Generalized Mean Value Theorem in 3.8 gives $a < c < c_2 < c_1 < x$

25 $1 + \frac{1}{2}x^2 + \frac{3}{8}(x^2)^2$ 27 $(-1)^n; (-1)^n(n+1)$

29 (a) one friend k times, the other $n - k$ times, $0 \le k \le n$; 21 33 $(16-1)^{1/4} \approx 1.968$

35 $(1 + .1)^{1.1} = 1(1.1)(.1) + \frac{(1.1)(.1)}{2}(.1)^2 \approx 1.1105$ 37 $1 + \frac{x^2}{2} + \frac{5x^4}{24}; r = \frac{\pi}{2}$ 41 $x + x^2 + \frac{5}{6}x^3 + \frac{5}{6}x^4$

43 $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6$ 45 $1 + \frac{x}{2} + \frac{3x}{8} + \frac{5x}{16}$ 47 .2727 49 $-\frac{1}{6} - \frac{1}{3} = -\frac{1}{2}$ 51 $r = 1, r = \frac{\pi}{2} - 1$

2 In the geometric series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ change x to $4x^2$: $\frac{1}{1-4x^2} = \mathbf{1 + 4x^2 + 16x^4} + \cdots$. Convergence fails when $4x^2$ reaches 1(thus $\mathbf{x = \frac{1}{2}}$ or $\mathbf{x = -\frac{1}{2}}$). The radius of convergence is $\mathbf{r = \frac{1}{2}}$.

4 $\tan x$ has derivatives $\sec^2 x, 2\sec^2 x \tan x, 2\sec^4 x + 4\sec^2 x \tan^2 x$. At $x = 0$ the series is $1 + 0x + \frac{2}{2}x^2 + 0x^3 = \mathbf{1 + x^2}$. The function $\tan x = \frac{\sin x}{\cos x}$ is infinite when $\cos x = 0$, at $\mathbf{x = \frac{\pi}{2}}$ and $\mathbf{x = -\frac{\pi}{2}}$. Then $\mathbf{r = \frac{\pi}{2}}$.

6 In the geometric series replace x by $-4x^2$. Then $\frac{1}{1+4x^2} = \mathbf{1 - 4x^2 + 16x^4} - \cdots$. Convergence fails when $|4x^2|$ reaches 1. The function blows up when $4x^2 = -1$, at $\mathbf{x = \frac{i}{2}}$ and $\mathbf{x = -\frac{i}{2}}$. The radius of convergence is $\mathbf{r = \frac{1}{2}}$.

8 The derivative of $\sum(x-a)^n = \frac{1}{1-(x-a)}$ is $\sum n(x-a)^{n-1} = \mathbf{\frac{1}{(1-x+a)^2}}$. The first series converges between $\mathbf{x = a - 1}$ and $\mathbf{x = a + 1}$. The derivative has the same interval of convergence. The series do not converge (the terms don't approach zero) at the endpoints $x = a - 1$ and $x = a + 1$.

10 $(x - 2\pi) - \frac{(x-2\pi)^3}{3!}$ begins the Taylor series for $\sin(x - 2\pi) = \sin x$, with basepoint $a = 2\pi$. The series converges for all x (thus $\mathbf{r = \infty}$) because of the factorials $3!, 5!, 7!, \cdots$.

12 $xe^x = x(1 + x + \cdots + \frac{x^n}{n!} + \cdots) = x + x^2 + \cdots + \frac{x^{n+1}}{n!} + \cdots$. Integrate the function and its series from 0 to 1: $\int_0^1 xe^x dx = [xe^x - e^x]_0^1 = \mathbf{1} = \int_0^1 (x + x^2 + \cdots + \frac{x^{n+1}}{n!} + \cdots)dx = \mathbf{\frac{1}{2} + \frac{1}{3}} + \cdots + \mathbf{\frac{1}{n!(n+2)}} + \cdots$.

14 (a) Combine $x + x^4 + x^7 + \cdots = \frac{x}{1-x^3}$ and $x^2 + x^5 + x^8 + \cdots = \frac{x^2}{1-x^3}$ and $-(x^3 + x^6 + \cdots) = -\frac{x^3}{1-x^3}$ to get $\mathbf{\frac{x+x^2-x^3}{1-x^3}}$. (b) Adding the series for $\cos x$ and $\cosh x$ leads to $1 + \frac{x^4}{4!} + \frac{x^8}{8!} + \cdots = \mathbf{\frac{1}{2}(\cos x + \cosh x)}$. (c) $\ln(x-1) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \cdots$ so changing x to $x - 1$ gives the series for $\mathbf{\ln(x-2)}$ around $a = 1$.

16 $\mathbf{\sum(x-\pi)^n}$ converges for $0 < x < 2\pi$ (to the function $\frac{1}{1-(x-\pi)}$).

18 The first missing term in the sine series is $\frac{(x-2\pi)^5}{5!}$. In equation (2) for the remainder $R_4(x)$, the derivative $f^{(5)} = \cos x$ is evaluated at some point c instead of at 2π. Always $|\cos c| \le 1$ so the error is less than $\frac{|x-2\pi|^5}{5!}$. (Confirmed by the alternating series rule: error less than first omitted term.)

20 For the function $f(x) = -\ln(1-x)$ with $f' = \frac{1}{1-x}, f'' = \frac{1}{(1-x)^2}, f''' = \frac{2}{(1-x)^3}$, the error after these terms is $|R_3(x)| \le f''''(c)\frac{x^4}{24} = \frac{6}{(1-c)^4}\frac{(\frac{1}{2})^4}{24} \le \frac{1}{4}$ (instead of $\frac{1}{8}$: set $c = 0$). A direct estimate of the missing terms in the series is $R_4 \le \frac{(\frac{1}{2})^4}{4} + \frac{(\frac{1}{2})^5}{5} + \cdots \le \frac{1}{4}((\frac{1}{2})^4 + (\frac{1}{2})^5 + \cdots) = \frac{1}{32}$.

22 The remainder after n terms of the series for e^x around $a = 1$ is $R_n(x) = e^c\frac{(x-1)^{n+1}}{(n+1)!}$. The factor e^c is between 1 and e^x. As $n \to \infty$ the factorial assures that $R_n(x) \to 0$ and the series converges to e^x.

24 $f(x) = e^{-1/x^2}$ equals $\mathbf{e^{-100}}$ at $x = .1$. However, the Taylor series is identically zero: $0 + 0x + 0x^2 + \cdots$. The radius of convergence is $\mathbf{r = \infty}$ but the series agrees with $f(x)$ only at $x = 0$. The error at $x = 1$ in linear approximation ($n = 1$) is $|R_1(1)| \le f''(c)\frac{(.1)^2}{2} = \frac{1}{200}(\frac{4}{c^6} - \frac{6}{c^4})e^{-1/c^2}$. Certainly the difference

between e^{-1/x^2} and $0 + 0x$ is $\mathbf{e^{-1}}$ at $x = 1$.

26 The derivatives of $(1-x)^{-1/2}$ are $\frac{1}{2}(1-x)^{-3/2}, \frac{1\cdot 3}{2\cdot 2}(1-x)^{-5/2}, \cdots, \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}(1-x)^{-(2n+1)/2}$. At $x = 0$ this nth derivative divided by $n!$ is the coefficient a_n.

28 $\sum_{n=1}^{\infty} nx^{n-1} =$ (with $m = n-1$) $\cdot \sum_{m=0}^{\infty}(m+1)x^m =$ (with m replaced by n) $\sum_{n=0}^{\infty}(n+1)x^n$.

30 (a) $(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots) = 1 + 2x + 3x^2 + \cdots$. The coefficient of x^n is $\mathbf{n+1}$.

(b) Multiply again by $1 + x + x^2 + \cdots$ to get $1 + 3x + 6x^2 + \cdots$. This is $(\frac{1}{1-x})^3 =$ cube of geometric series for $\frac{1}{1-x}$. The derivatives are $\frac{3}{(1-x)^4}, \frac{4\cdot 3}{(1-x)^5}, \frac{5\cdot 4\cdot 3}{(1-x)^6}, \frac{6\cdot 5\cdot 4\cdot 3}{(1-x)^7}, \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{(1-x)^8}$. The coefficient of x^5 is the 5th derivative at $x = 0$ divided by $5! = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \mathbf{21}$.

32 This is Problem 26 with x changed to $4x$. So the coefficient of x^n is multiplied by 4^n. By Problem 26 this gives $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}4^n = \frac{1\cdot 3\cdots(2n-1)}{n!}\frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 2\cdot 3\cdots n} = \frac{(2n)!}{(n!)^2}$.

34 Take $p = \frac{1}{3}$ and $x = .001$; the binomial series gives $(1.001)^{1/3}$ and multiply by 10 to get $(1001)^{1/3} = 10[1 + \frac{1}{3}(.001) - \frac{1}{9}(.001)^2 \cdots] = 10.003\cdots$.

36 Take $p = \frac{1}{1000}$ and $x = e - 1$: the binomial series is $e^p = (1+x)^p = 1 + \frac{e-1}{1000} + \cdots = 1 + .0018 + \cdots$ which diverges since $x > 1$!! The ordinary series $e^p = 1 + p + \frac{1}{2}p^2 + \cdots$ correctly gives $e^{1/1000} = 1.0010005\cdots$.

38 $\sec^2 x = \frac{1}{1-\sin^2 x} \approx \frac{1}{1-x^2} \approx \mathbf{1 + x^2}$. Check by squaring in Problem 37: $(\sec x)^2 = (1 + \frac{x^2}{2} + \cdots)^2 \approx 1 + x^2$. Check by derivative of $\tan x = x + \frac{x^3}{3} + \cdots$ to find $1 + x^2 + \cdots$.

40 $f(g(x)) \approx a_0 + a_1(b_1x + b_2x^2 + \cdots) + a_2(b_1x + b_2x^2 + \cdots)^2 \approx \mathbf{a_0 + a_1b_1x + (a_1b_2 + a_2b_1^2)x^2}$. Test on $f = \frac{1}{1+x} \approx 1 - x + x^2$ (which has $a_0 = 1, a_1 = -1, a_2 = 1$) and $g = \frac{x}{1-x} \approx x + x^2$ (which has $b_1 = 1 = b_2$). The formula correctly gives $f(g(x)) = 1 - x + (0)x^2$.

42 By Problem 40 with $a_0 = 0$ the series starts with $f(g(x)) = a_1b_1x + (a_1b_2 + a_2b_1^2)x^2$. This agrees with $f(g(x)) = x$ when $b_1 = \frac{1}{a_1}$ and $b_2 = -\frac{a_2b_1^2}{a_1} = -\frac{a_2}{a_1^3}$. The example $f = e^x - 1 = x + \frac{x^2}{2!} + \cdots$ has $a_1 = 1$ and $a_2 = \frac{1}{2}$ so that $b_1 = \frac{1}{1}$ and $b_2 = \frac{-\frac{1}{2}}{1^3}$. These are the coefficients in $f^{-1}(x) = \ln(1+x) = x - \frac{x^2}{2} + \cdots$.

44 Quick method: Multiply $(1-x)(1 + x^3 + x^6 + \cdots) = 1 - x + x^3 - x^4 + x^6 - x^7 + \cdots$.

Slow method: $\frac{1-x}{1-x^3} = \frac{1}{1+x+x^2} =$ (geometric series for $-x - x^2$) $= 1 - x - x^2 + (x+x^2)^2 - (x+x^2)^3 + (x+x^2)^4 - (x+x^2)^5 \approx 1 - x + 0x^2 + x^3 - x^4 + 0x^5$.

46 $\int_0^1 e^{-x^2}dx \approx \int_0^1 (1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120})dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2} - \frac{1}{7\cdot 6} + \frac{1}{9\cdot 24} - \frac{1}{11\cdot 120} = .747$ to 3 places.

48 At $x = -1$ the alternating series $\sum \frac{x^n}{n} = \sum \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \cdots$ converges (to $\ln(1-x) = \ln 2$). The derivative $\sum x^{n-1} = 1 + x + x^2 + \cdots = 1 - 1 + 1 - \cdots$ diverges. Both series have $\mathbf{r = 1}$; one series converges at an endpoint of the interval $-1 < x < 1$ and the other doesn't.

50 If $a_n^{1/n}$ approaches L then $(a_nx^n)^{1/n}$ approaches $\frac{x}{L}$. By the root test the series $\sum a_nx^n$ converges when $|\frac{x}{L}| < 1$ and diverges when $|\frac{x}{L}| > 1$. So the radius of convergence is $r = L$.