CHAPTER 1 INTRODUCTION TO CALCULUS

1.1 Velocity and Distance (page 6)

Starting from $f(0) = 0$ at constant velocity v, the distance function is $f(t) = \mathbf{vt}$. When $f(t) = 55t$ the velocity is $v = \mathbf{55}$. When $f(t) = 55t + 1000$ the velocity is still **55** and the starting value is $f(0) = \mathbf{1000}$. In each case v is the **slope of the graph of** f. When $\mathbf{v(t)}$ is negative, the graph of $\mathbf{f(t)}$ **goes downward**. In that case area in the v–graph counts as **negative**.

Forward motion from $f(0) = 0$ to $f(2) = 10$ has $v = \mathbf{5}$. Then backward motion to $f(4) = 0$ has $v = \mathbf{-5}$. The distance function is $f(t) = 5t$ for $0 \le t \le 2$ and then $f(t)$ equals $\mathbf{5(4-t)}$ (not $-5t$). The slopes are **5** and **−5**. The distance $f(3) = \mathbf{5}$. The area under the v–graph up to time 1.5 is **7.5**. The domain of f is the time interval $\mathbf{0 \le t \le 4}$, and the range is the distance interval $\mathbf{0 \le f \le 10}$. The range of $v(t)$ is only **5 and −5**.

The value of $f(t) = 3t + 1$ at $t = 2$ is $f(2) = \mathbf{7}$. The value 19 equals $f(\mathbf{6})$. The difference $f(4) - f(1) = \mathbf{9}$. That is the change in distance, when $4 - 1$ is the change in **time**. The ratio of those changes equals **3**, which is the **slope of the graph**. The formula for $f(t) + 2$ is $3t + 3$ whereas $f(t + 2)$ equals $\mathbf{3t+7}$. Those functions have the same **slope** as f : the graph of $f(t) + 2$ is shifted **up** and $f(t + 2)$ is shifted **to the left**. The formula for $f(5t)$ is $\mathbf{15t+1}$. The formula for $5f(t)$ is $\mathbf{15t + 5}$. The slope has jumped from 3 to **15**.

The set of inputs to a function is its **domain**. The set of outputs is its **range**. The functions $f(t) = 7+3(t-2)$ and $f(t) = vt + C$ are **linear**. Their graphs are **straight lines** with slopes equal to **3** and **v**. They are the same function, if $v = \mathbf{3}$ and $C = \mathbf{1}$.

1 $v = 30, 0, -30; v = -10, 20$ **3** $v(t) = \begin{cases} 2 \text{ for } 0 < t < 10 \\ 1 \text{ for } 10 < t < 20 \\ -3 \text{ for } 20 < t < 30 \end{cases}$ $v(t) = \begin{cases} 0 & \text{for } 0 < t < T \\ \frac{1}{T} & \text{for } T < t < 2T \\ 0 & \text{for } 2T < t < 3T \end{cases}$

5 $25; 22; t + 10$ **7** $6; -30$ **9** $v(t) = \begin{cases} 20 \text{ for } t < .2 \\ 0 \text{ for } t > .2 \end{cases}$ $f(t) = \begin{cases} 20t & \text{for } t \le .2 \\ 4 & \text{for } t \ge .2 \end{cases}$ **11** $10\%; 12\frac{1}{2}\%$

13 $f(t) = 0, 30(t-1), 30; f(t) = -30t, -60, 30(t-6)$ **15** Average 8, 20 **17** $40t - 80$ for $1 \le t \le 2.5$

21 $0 \le t \le 3, -40 \le f \le 20; 0 \le t \le 3T, 0 \le f \le 60T$ **23** $3 - 7t$ **25** $6t - 2$ **27** $3t + 7$

29 Slope -2; $1 \le f \le 9$ **31** $v(t) = \begin{cases} 8 & \text{for } 0 < t < T \\ -2 & \text{for } T < t < 5T \end{cases}$ $f(t) = \begin{cases} 8t & \text{for } 0 \le t \le T \\ 10T - 2t & \text{for } T \le t \le 5T \end{cases}$

33 $\frac{9}{5}C + 32$; slope $\frac{9}{5}$ **35** $f(w) = \frac{w}{1000}$; slope = conversion factor **37** $1 \le t \le 5, 0 \le f \le 2$

39 $0 \le t \le 5, 0 \le f \le 4$ **41** $0 \le t \le 5, 1 \le t \le 32$ **43** $\frac{1}{2}t + 4; \frac{1}{2}t + \frac{7}{2}; 2t + 12; 2t + 3$

45 Domains $-1 \le t \le 1$: ranges $0 \le 2t + 2 \le 4$, $-3 \le t - 2 \le -1$, $-2 \le -f(t) \le 0$, $0 \le f(-t) \le 2$

47 $\frac{3}{2}V; \frac{3}{2}V$ **49** input * input → A; input +A → output | input * input → A; input +A → B | B * B → C; B + C → output | input +1 → A; A * A → B; A + B → output

51 $3t + 5, 3t + 1, 6t - 2, 6t - 1, -3t - 1, 9t - 4$; slopes 3,3,6,6,−3,9

53 The graph goes up and down *twice*. $f(f(t)) = \begin{cases} 2(2t) & 0 \le t \le 1.5 \\ 12 - 4t & 1.5 \le t \le 3 \\ 12 - 2(12 - 2t) & 3 \le t \le 4.5 \\ 2(12 - 2t) & 4.5 \le t \le 6 \end{cases}$

2 (a) The slopes are $v = \mathbf{2}$ then $v = \mathbf{1}$ then $v = \mathbf{-3}$

(b) The slopes are $v = 0$ then $v = \mathbf{1/T}$ then $v = \mathbf{0}$

4 $f(t) = \mathbf{20(t-1)}$ for $1 \le t \le 2$

6 $f(1.4T) = .\mathbf{4}$; if $T = 3$ then $f(4) = \mathbf{\frac{1}{3}}$. This is $\frac{1}{3}$ of the distance between $f(3) = 0$ and $f(6) = 1$.

8 Average speed $= \frac{f(2)-f(0)}{2} = \frac{20-10}{2} = \mathbf{5}$; the average speed is zero between $t = \frac{1}{2}$ and $t = \mathbf{1\frac{1}{4}}$, since at both times $f = 5$.

10 $v(t)$ is negative-zero-positive; $v(t)$ is above 55 then equal to 55; $v(t)$ increases in jumps; $v(t)$ is zero then positive. All with corresponding $f(t)$.

12 $f(t)$ increases linearly from 5.2 billion in 1990 to **6.2 billion** in 2000.

14 (a) $f(t) = -40t$ (graph drops linearly to -40 at $t = 1$) then $f(t) = -40 + 40(t-1) = \mathbf{40t - 80}$. End at $f(\frac{5}{2}) = 20$

(b) Second graph rises to $40T$ at time T, stays constant until time $2T$, then rises more slowly to $60T$ at time $3T$.

16 $f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 30(t-1) & 1 \le t \le 2; \\ 30 & t \ge 2 \end{cases}$ $f(t) = \begin{cases} -30t & 0 \le t \le 2 \\ -60 & 2 \le t \le 4 \\ -60 + 30(t-4) & t > 4 \end{cases}$

18 $v(t) = 8$ then $\mathbf{1}$ (after $t = 2$); $f(t) = \mathbf{6 + 8t}$ then $20 + t$.

20 $1200 + 30x = 40x$ when $1200 = 10x$ or $x = \mathbf{120}$ yearbooks. The slope is **30.** If it goes above **40** you can't break even.

22 Range $= \{\mathbf{0, 20, 40}\}$; the velocity is not defined at the jump.

24 $f(t) = \mathbf{4t + 1}$ (linear up) or $\mathbf{-4t + 9}$ (linear down).

26 The function increases by 2 in one time unit so the slope (velocity) is 2; $f(t) = \mathbf{2t + C}$ with constant $C = f(0)$.

28 $f(2t) = 2vt$ must equal $4vt$ so $v = 0$ and **f=0.** But $\frac{1}{2}a(2t)^2$ does equal $4(\frac{1}{2}at^2)$. To go **four** times as far in twice the time, you must accelerate.

30 $f(t) = 0$ then $\mathbf{8 - 2t}$ (change at $t = 4$); slopes 0 and -2; range $\mathbf{-2} \le f(t) \le \mathbf{0}$.

32 $f(t) = 3t = 12$ at $t = 4$; then $v = \mathbf{6}$ gives $f(t) = 12 + 6(t-4) = 30$ at $t = 7$. The extra distance was 18 in 3 time units; thus $v(t) = 3$ then 6.

34 $C(F) = \mathbf{\frac{5}{9}(F - 32)}$ has slope $\mathbf{\frac{5}{9}}$.

36 At $t = 0$ the reading was $.061 + 10(.015) = \mathbf{.211}$. A drop of $.061 - .04 = .021$ would take $\mathbf{.021/.015}$ hours. This was the Exxon Valdez accident.

38 Domain $1 < t \le 5$; range $\frac{1}{4} \le f(t) < \infty$.

40 Domain $0 \le t < 4$ and $4 < t \le 5$ (**omit** $\mathbf{t = 4}$); range $\frac{1}{16} \le f(t) < \infty$

42 Domain $0 \le t \le 5$; range 2^{-5} (or $\frac{1}{32}$) $\le f(t) \le 1$.

44 Jump from 0 to 1 at $t = 0$; jump from **2** to **3** at $t = 0$; jump from 0 to 1 at $t = \mathbf{-2}$; jump from 0 to **3** at $t = 0$; jump from 0 to **1** at $t = 0$.

46 $2f(3t) = 2(3t-1) = \mathbf{6t - 2}$; $f(1-t) = (1-t) - 1 = \mathbf{-t}$; $f(t-1) = (t-1) - 1 = \mathbf{t - 2}$.

48 $f_1(t) = \mathbf{3t + 3}$; $f_2(t) = \mathbf{3t + 18}$.

50 "A function assigns an output to each input ····"

52 $3(vt + C) + 1$ has slope $\mathbf{3v}$; $v(3t + 1) + C$ also has slope $\mathbf{3v}$; $2(4vt + C)$ has slope $\mathbf{8v}$; $-vt + C$ has slope $\mathbf{-v}$; $vt + C - C$ has slope $\mathbf{v}$; $v(vt + C) + C$ has slope $\mathbf{v^2}$.

54 A function cannot have two values (the upper and lower branches of **X**) at the same point. Apparently only

U, V, W are graphs. Their slopes are negative-positive and negative-positive-negative-positive.

1.2 Jumps in Velocity (page 14)

When the velocity jumps from v_1 to v_2, the function $v(t)$ is piecewise **constant**. The distance function $f(t)$ is piecewise **linear**. In the first time interval, $f(t) = f(0) + \mathbf{v_1 t}$. After the jump at $t = 1$, the formula is $f(t) = f(1) + \mathbf{v_2(t-1)}$. In case $f_0 = 6$ all distances are increased by **6** and all velocities are **the same**.

With distances 1, 5, 25 at unit times, the velocities are **4 and 20**. These are the **slopes** of the f-graph. The slope of the tax graph is the tax **rate**. If $f(t)$ is the postage cost for t ounces or t grams, the slope is the **cost per ounce (or per gram)**. For distances 0, 1, 4, 9 the velocities are **1, 3, 5**. The sum of the first j odd numbers is $f_j = \mathbf{j^2}$. Then f_{10} is **100** and the velocity v_{10} is **19**.

The piecewise linear sine has slopes **1, 0, −1, −1, 0, 1**. Those form a piecewise **constant** cosine. Both functions have **period** equal to 6, which means that $f(t+6) = \mathbf{f(t)}$ for every t. The velocities $v = 1, 2, 4, 8, \ldots$ have $v_j = \mathbf{2^{j-1}}$. In that case $f_0 = 1$ and $f_j = \mathbf{2^j}$. The sum of 1, 2, 4, 8, 16 is **31**. The difference $2^j - 2^{j-1}$ equals $\mathbf{2^{j-1}}$. After a burst of speed V to time T, the distance is **VT**. If $f(T) = 1$ and V increases, the burst lasts only to $T = \mathbf{1/V}$. When V approaches infinity, $f(t)$ approaches a **step** function. The velocities approach a **delta** function, which is concentrated at $t = 0$ but has area **1** under its graph. The slope of a step function is **zero or infinity**.

1 1.1, −2, 5 **3** 6.6, 8.8; −11, −15; 4, 14 **5** $h(t) = 9t + 6$, add slopes **7** $f = 2t$ then $3t - T$
9 7, 28, $8t + 4$; multiply slopes **11** 16, 0, $8t$ then $36 - 4t$ **13** Tax $= .28x$; 280,000 **15** $19\frac{1}{4}\%$
17 All $v_j = 2$; $v_j = (-1)^{j-1}$; $v_j = (\frac{1}{2})^j$ **21** $j^2 + j$ **23** $f_{10} = 38$ **25** $(101^2 - 99^2)/2 = \frac{400}{2}$
27 $v_j = 2j$ **29** $f_{31} = 5$ **31** $a_j = -f_j$ **33** 0; 1; .1 **35** require $v_2 = -v_1$
37 $v_j = 3(4)^{j-1}$ **39** $v_j = -(\frac{1}{2})^j$ **41** $v_j = 2(-1)^j$, sum is $f_j - 1$ **45** $v = 1000, t = 10/V$
47 M, N **51** $\sqrt{9} < 2 \cdot 9 < 9^2 < 2^9; (\frac{1}{9})^2 < 2(\frac{1}{9}) < \sqrt{1/9} < 2^{1/9}$

2 $f(6), f(7)$ are 66, 77 and −11, −13 and 4, 9. Then $f(7) - f(6)$ is **11, −2, 5**.
4 The increases $f(4) - f(1)$ are $12 - 3 = \mathbf{9}$ and $14 - 5 = \mathbf{9}$ and $18 - 9 = \mathbf{9}$.
6 $h(t) = \mathbf{.5t + 3}$; the slopes of f, g, h are **3, 2.5** and $3 - 2.5 = \mathbf{.5}$.
8 $f(t) = \mathbf{1 + 10t}$ for $0 \le t \le \frac{1}{10}$, $f(t) = \mathbf{2}$ for $t \ge \frac{1}{10}$
10 $f(3) = \mathbf{12}$; $g(f(3)) = g(12) = \mathbf{25}$; $g(f(t)) = g(4t) = \mathbf{8t + 1}$. Distance increases **four** times as fast and velocity is multiplied by **4**.
12 10,160.50 is $f(44{,}900) = 2782.50 + .28(44{,}900 - 18{,}550)$.
14 $F(x) = \mathbf{2f(\frac{1}{2}x)} = .15x$ for $x \le 37{,}100$; then $F(x) = 5565 + .28(x - 37{,}100)$ up to $x = 89{,}800$; then $F(x) = 20{,}321 + .33\ (x - 89{,}800)$ up to $x = 186{,}260$; then $F(x) = .28x$ beyond 186,260. **The 1991 rates on the front cover have only three brackets.**
16 $f(t) = \mathbf{3 + 2t}$ for $t \ge 1$ is continuous; $f(t) = \mathbf{4 + 2t}$ is discontinuous (because $f(1) = 5$). $f(x) = .15x$

then **3000** + .28 $(x - 18,550)$ has a jump at \$18,550.

18 $f_1 = 1, f_2 = 3, f_3 = 7, f_j = \mathbf{2^j - 1}; f_1 = -1, f_2 = 0, f_3 = -1, f_j = \{-1 \text{ for odd } j, 0 \text{ for even } j\}$ $= \frac{1}{2}((-1)^j - 1)$.

20 The big triangle has area $= \frac{1}{2}$ (base)(height) $= \frac{1}{2}j^2$ and the j small triangles have area $\frac{1}{2}j$. Together they give rectangles of total area $1 + 2 + \cdots + j$. Note: Another drawing could move the diagonal line up by $\frac{1}{2}$. The big triangle still has area $\frac{1}{2}j^2$ and the strip across the bottom has area $\frac{1}{2}j$.

22 **False** when the v_j are $(\frac{1}{2})^j$; **false** when the v_j are $-(\frac{1}{2})^j$; **true** when all $f_{j+p} = f_j$ (p is the period) because then $v_{j+p} = f_{j+p} - f_{j+p-1}$ equals $f_j - f_{j-1} = v_j$; **false** when all $v_j = 1$.

24 Assume $f_0 = 0$. First $f_j = \mathbf{j^2}$, second $f_j = \mathbf{j}$, by addition third $f_j = \mathbf{j^2 + j}$, by division last $f_j = \frac{1}{2}(j^2 + j)$ which is $1 + 2 + \cdots + j$.

26 $f(99) = 9900$ and $f(101) = 10302; \Delta f/\Delta t = 402/2 = \mathbf{201}$.

28 Take $v = C, 2C, 3C, \cdots$ Then $f = C, 3C, 6C, \ldots$. The example $f_3 - 2f_2 + f_1$ gives $6C - 2(3C) + C = \mathbf{C}$. The answer is always **C** (by Problem 30).

30 $f_{j+1} - 2f_j + f_{j-1}$ equals $(f_{j+1} - f_j) - (f_j - f_{j-1}) = v_{j+1} - v_j$. If v is velocity then a is **acceleration**.

32 The period of $v + w$ is **30**, the smallest multiple of both 6 and 10. (Then v completes five cycles and w completes three.) An example for functions is $v = \sin \frac{\pi x}{3}$ and $w = \sin \frac{\pi x}{5}$ ($v + w$ has a nice graph).

34 $f(12) = (1+2+1+0) + (1+2+1+0) + (1+2+1+0) = \mathbf{12}$. Then $f(14) = 12 + 1 + 2 = \mathbf{15}$ and $f(16) = 15 + 1 + 0 = \mathbf{16}$. f doesn't have period 4 since $x_1 + x_2 + x_3 + x_4$ is **not zero**.

36 2^j is 2 times 2^{j-1}. Subtracting 2^{j-1} leaves 2^{j-1}. Similarly 3^j is 3 times 3^{j-1} and subtraction leaves **2 times $\mathbf{3^{j-1}}$**.

38 $f_1 - f_0$ equals $v_1 = 2f_0 = 2$ so f_1 is 3; $f_2 - f_1$ equals $v_2 = 2f_1 = 6$ so f_2 is 9; then f_3 is 27 and f_4 is **81**. Problem 36 shows that $f_j = 3^j$ fits the requirement $v_j = 2f_{j-1}$.

40 $v_j = f_j - f_{j-1}$ equals $\mathbf{r^j - r^{j-1}}$. Adding the v's gives $(f_1 - f_0) + (f_2 - f_1) + (f_3 - f_2) + \cdots + (f_j - f_{j-1})$. Cancelling leaves only $f_j - f_0 = \mathbf{r^j - 1}$.

42 The first sum is $1024 - 1 = \mathbf{1023}$. The second is $2 - \frac{1}{512} = \mathbf{\frac{1023}{512}}$. (Notice how the second sum is $\frac{1}{512}$ times the first.) The sum formula is in Problem 43 and also Problem 18.

44 $U(t) - U(t-1)$ is **zero** except between $t = 0$ and $t = 1$ (where it equals **1**). If this is the velocity, then the distance is $\mathbf{f(t) = t}$ up to $t = 1$; then $\mathbf{f(t) = 1}$: a "short burst of speed". If the square wave is distance, then $v(t)$ is a delta function at $t = 0$ **minus** a delta function at $t = 1$.

46 The sum jumps up by 1 at **t = 0, 1, 2**. Its slope is a sum of three **delta** functions.

48 $\begin{cases} \text{For } j = 1, N \text{ do} \\ v_j = f_j - f_{j-1} \end{cases}$ Examples $2j$ and j^2 and 2^j give $v_j = 2$ and $v_j = 2j - 1$ and $v_j = 2^{j-1}$.

50 FINDV (FINDF $(v_1, \cdots v_N)$) brings back $v_1, \cdots, v_N$. But FINDF (FINDV $(f_0, f_1, \cdots, f_N)$) produces $0, f_1 - f_0, f_2 - f_0, \cdots, f_N - f_0$.

52 The average age increases with slope 1 except at a birth or death (when it is discontinuous).

1.3 The Velocity at an Instant (page 21)

Between the distances $f(2) = 100$ and $f(6) = 200$, the average velocity is **25**. If $f(t) = \frac{1}{4}t^2$ then $f(6) = 9$ and $f(8) = \mathbf{16}$. The average velocity in between in **3.5**. The instantaneous velocities at $t = 6$ and $t = 8$ are **3** and **4.**

The average velocity is computed from $f(t)$ and $f(t+h)$ by $v_{ave} = \mathbf{\frac{1}{h}(f(t+h)-f(t))}$. If $f(t) = t^2$ then $v_{ave} = \mathbf{2t+h}$. From $t=1$ to $t=1.1$ the average is **2.1.** The instantaneous velocity is the **limit** of v_{ave}. If the distance is $f(t) = \frac{1}{2}at^2$ then the velocity is $v(t) = \mathbf{at}$ and the acceleration is **a.**

On the graph of $f(t)$, the average velocity between A and B is the slope of **the secant line.** The velocity at A is found by **letting B approach A.** The velocity at B is found by **letting A approach B.** When the velocity is positive, the distance is **increasing.** When the velocity is increasing, the car is **accelerating.**

1 $6, 6, \frac{13}{2}a, -12, 0, 13$ **3** 4, 3.1, $3+h$, 2.9 **5** Velocity at $t=1$ is 3 **7** Area $f = t+t^2$, slope of f is $1+2t$
9 F; F; F; T **11** $2; 2t$ **13** $12+10t^2; 2+10t^2$ **15** Time 2, height 1, stays above $\frac{3}{4}$ from $t=\frac{1}{2}$ to $\frac{3}{2}$
17 $f(6) = 18$ **21** $v(t) = -2t$ then $2t$ **23** Average to $t=5$ is 2; $v(5) = 7$ **25** $4v(4t)$ **27** $v_{ave} = t, v(t) = 2t$

2 (a) $\frac{6(t+h)-6t}{h} = 6$ (limit is 6); (b) $\frac{6(t+h)+2-(6t+2)}{h} = 6$ (limit also 6); (c) $\frac{\frac{1}{2}a(t^2+2th+h^2)-\frac{1}{2}at^2}{h} = at+\frac{1}{2}ah$ (limit is at); (d) $\frac{t+h-(t+h)^2-(t-t)^2}{h} = 1-2t-h$ (limit is $1-2t$); (e) $\frac{6-6}{h} = 0$ (limit is 0); (f) the limit is $v(t) = 2t$ (and $f(t) = t^2$ gives $\frac{(t+h)^2-t^2}{h} = 2t+h$).
4 $\frac{\Delta f}{\Delta t} = \frac{2-0}{1} = 2; \frac{3/4-0}{1/2} = \frac{3}{2}; \frac{h+h^2-0}{h} = 1+h$. **6** $\lim \frac{\Delta f}{\Delta t} = \lim(1+h) = 1 =$ slope of the parabola at $t=0$.
8 $v(t) = 3-2t$ gives a line through (0,3) and (1,1); $f(t) = 3t-t^2$ gives a parabola through (0,0) and (3,0) with maximum at $(\frac{3}{2}, \frac{9}{4})$.
10 Slope of $f(t) = 6t^2$ is $v(t) = 12t$; slope of $v(t) = 12t$ is $a = 12 =$ acceleration.
12 $\Delta f = \frac{1}{2}a(t+h)^2 - \frac{1}{2}a(t-h)^2 = 2\,ath$; then $\frac{\Delta f}{\Delta t} = \frac{2\,ath}{2h} = at =$ velocity at time t. The region under the line $v = at$ is a trapezoid. Its area is the base $2h$ times the average height at.
14 True (the slope is $\frac{\Delta f}{\Delta t}$); false (the curve is partly steeper and partly flatter than the secant line which gives the average slope); true (because $\Delta f = \Delta F$); false (V could be larger than v in between).
16 The functions are t^2 and t^2-2 and $4t^2$. The velocities are $2t$ and $2t$ and $8t$.
18 The graph is a parabola $f(t) = \frac{1}{2}t^2$ out to $f = 2$ at $t = 2$. After that the slope of f stays constant at 2.
20 Area to $t=1$ is $\frac{1}{2}$; to $t=2$ is $\frac{3}{2}$; to $t=3$ is 2; to $t=4$ is $\frac{3}{2}$; to $t=5$ is $\frac{1}{2}$; area from $t=0$ to $t=6$ is zero. The graph of $f(t)$ through these points is parabola-line-parabola (symmetric)-line-parabola to zero.
22 $f(t)$ is a parabola $t-\frac{1}{2}t^2$ through (0,0), $(1,\frac{1}{2})$, and (2,0); $f(t)$ is the same parabola until $(1,\frac{1}{2})$, but the second half goes up to (2,1); $f(t)$ is the parabola $2t-t^2$ until (1,1) and then a horizontal line since $v=0$.
24 The slope of f is $v(t) = at+b$; the slope of v is the constant a; $f(t) = \frac{1}{2}t^2+t+1$ equals 41 when $t=8$. (The quadratic formula for $\frac{1}{2}t^2+t-40=0$ gives $t = -1 \pm \sqrt{1^2+80} = -1 \pm 9$.)
26 $f(t) = t-t^2$ has $v(t) = 1-2t$ and $f(3t) = 3t-9t^2$. The slope of $f(3t)$ is $3-18t$. This is $\mathbf{3v(3t)}$.
28 To find $f(t)$ multiply the time t by the average velocity. This is because $v_{ave}(t) = \frac{f(t)-f(0)}{t} = \frac{f(t)}{t}$.

1.4 Circular Motion (page 28)

A ball at angle t on the unit circle has coordinates $x = \mathbf{\cos t}$ and $y = \mathbf{\sin t}$. It completes a full circle at $t = \mathbf{2\pi}$. Its speed is **1.** Its velocity points in the direction of the **tangent,** which is **perpendicular** to the radius coming out from the center. The upward velocity is **cos t** and the horizontal velocity is **− sin t**.

A mass going up and down level with the ball has height $f(t) =$ **sin t**. This is called simple **harmonic** motion. The velocity is $v(t) =$ **cos t**. When $t = \pi/2$ the height is $f =$ **1** and the velocity is $v =$ **0**. If a speeded-up mass reaches $f = \sin 2t$ at time t, its velocity is $v =$ **2 cos 2t**. A shadow traveling *under* the ball has $f = \cos t$ and $v =$ **−sin t**. When f is distance = area = integral, v is **velocity = slope = derivative**.

1 $10\pi, (0,-1), (-1,0)$ **3** $(4\cos t, 4\sin t); 4$ and $4t; 4\cos t$ and $-4\sin t$
5 $3t; (\cos 3t, \sin 3t); -3\sin 3t$ and $3\cos 3t$ **7** $x = \cos t; \sqrt{2}/2; -\sqrt{2}/2$ **9** $2\pi/3; 1; 2\pi$
11 Clockwise starting at (1,0) **13** Speed $\frac{2}{\pi}$ **15** Area 2 **17** Area 0
19 4 from speed, 4 from angle **21** $\frac{1}{4}$ from radius times 4 from angle gives 1 in velocity
23 Slope $\frac{1}{2}$; average $(1 - \frac{\sqrt{3}}{2})/(\pi/6) = \frac{3(2-\sqrt{3})}{\pi} = .256$ **25** Clockwise with radius 1 from (1,0), speed 3
27 Clockwise with radius 5 from (0,5), speed 10 **29** Counterclockwise with radius 1 from (cos 1, sin 1), speed 1
31 Left and right from (1,0) to (-1,0), $v = -\sin t$ **33** Up and down between 2 and −2; start $2\sin\theta, v = 2\cos(t+\theta)$
35 Up and down from (0,−2) to (0,2); $v = \sin \frac{1}{2}t$ **37** $x = \cos\frac{2\pi t}{360}, y = \sin\frac{2\pi t}{360}$, speed $\frac{2\pi}{360}, v_{\text{up}} = \cos\frac{2\pi t}{360}$
39 I think there *is* a stop between backward and forward motion.

2 The cosine of $\frac{2\pi}{3}$ is $x = -\frac{1}{2}$; the sine is $y = \frac{\sqrt{3}}{2}$; the tangent is $\frac{y}{x} = -\sqrt{3}$; the ball has a distance $\sqrt{3}$ to go (draw triangle from (0,0) to (x,y) and back down at right angle); the speed is 1 so the added time is $\sqrt{3}$ and the total time is $\frac{2\pi}{3} + \sqrt{3}$. Not easy.
4 $x = R\cos t$ and $y = R\sin t$; velocity $-R\sin t$ and $R\cos t$; distance and velocity triangles both grow by R.
6 The angle is $\frac{\pi}{2} + 3t$; the position is $x = \cos(\frac{\pi}{2} + 3t) = -\sin 3t$ and $y = \sin(\frac{\pi}{2} + 3t) = \cos 3t$; the vertical velocity is $-3\sin 3t$ (= horizontal velocity of original ball).
8 The new mass at $x = \cos t, y = 0$ never meets the old mass at $x = 0, y = \sin t$. The distance between them is always $\sqrt{\cos^2 t + \sin^2 t} = 1$.
10 $f = \sin(t + \pi)$ equals $-\sin t$; the velocity is $\cos(t + \pi)$ which equals $-\cos t$. The ball is a half-circle ahead of the original ball.
12 $f(t) = \sin t + \cos t$ has $f^2 = \sin^2 t + 2\sin t\cos t + \cos^2 t$ which is the same as $1 + 2\sin t\cos t$ (or $1 + \sin 2t$). The maximum is at $t = 45° = \frac{\pi}{4}$ when $f^2 = 2$. Then $f_{\max} = \sqrt{2}$. Its graph is a sine curve with this maximum point: $f(t)$ equals $\sqrt{2}\sin(t + \frac{\pi}{4})$.
14 The ball goes halfway around the circle in time π. For the mass to fall a distance 2 in time π we need $2 = \frac{1}{2}a\pi^2$ so $a = 4/\pi^2$.
16 The area is $f(t) = \sin t$, and $\sin\frac{\pi}{6} - \sin 0 = \frac{1}{2}$.
18 The area is still $f(t) = \sin t$, and $\sin\frac{3\pi}{2} - \sin\frac{\pi}{2} = -1 - 1 = -2$.
20 The radius is 2 and time is speeded up by 3 so the velocity is 6 with minus sign because the cosine starts downward (ball moving to left).
22 The distance is $-\cos 5t$.
24 $\frac{\sin 1 - \sin 0}{1} = .8415$ and $\frac{\sin .1}{.1} = .9983$ and $\frac{\sin .01}{.01} = .9999$; then $\frac{\sin .001}{.001} = .99999983$.
26 Counterclockwise with radius 3 starting at (3,0) with speed 12.
28 Counterclockwise with radius 1 around center at (1,0). Starts from (2,0); speed 1.
30 Clockwise around the unit circle from (1,0) with speed 1.
32 Up and down between −1 and 1, starting at (0,0) with velocity 5 cos $5t$.
34 Along the 45° line $y = x$ between $(-1,-1)$ and (1,1). Starting at (1,1) with x and y velocities $-\sin t$.

36 Along the line $x+y=1$ between (1,0) and (0,1). Starting at (1,0) the x and y velocities are $-2\sin t\cos t$ and $2\sin t\cos t$. (Maybe introduce $\cos^2 t = \frac{1}{2}+\frac{1}{2}\cos 2t$ and $\sin^2 t = \frac{1}{2}-\frac{1}{2}\cos 2t$ to find velocities $-\sin 2t$ and $\sin 2t$: **Discuss.**)
38 Choose $k=2\pi$. The speed is 2π and the upward velocity is $2\pi\cos 2\pi t$.

1.5 Review of Trigonometry (page 33)

Starting with a **right** triangle, the six basic functions are the **ratios** of the sides. Two ratios (the cosine x/r and the **sine** y/r) are below 1. Two ratios (the secant r/x and the **cosecant r/y**) are above 1. Two ratios (the **tangent** and the **cotangent**) can take any value. The six functions are defined for all angles θ, by changing from a triangle to a **circle**.

The angle θ is measured in **radians**. A full circle is $\theta = 2\pi$, when the distance around is $2\pi r$. The distance to angle θ is θr. All six functions have period 2π. Going clockwise changes the sign of θ and $\sin\theta$ and $\tan\theta$. Since $\cos(-\theta)=\cos\theta$, the cosine is **unchanged (or even)**.

Coming from $x^2+y^2=r^2$ are the three identities $\sin^2\theta+\cos^2\theta=1$ and $\tan^2\theta+1=\sec^2\theta$ and $1+\cot^2\theta=\csc^2\theta$. (Divide by r^2 and x^2 and y^2.) The distance from (2,5) to (3,4) is $d=\sqrt{2}$. The distance from (1,0) to $(\cos(s-t),\sin(s-t))$ leads to the addition formula $\cos(s-t)= \cos s\cos t+\sin s\sin t$. Changing the sign of t gives $\cos(s+t)=\cos s\cos t-\sin s\sin t$. Choosing $s=t$ gives $\cos 2t=\cos^2 t-\sin^2 t$ or $2\cos^2 t-1$. Therefore $\frac{1}{2}(1+\cos 2t)=\cos^2 t$, a formula needed in calculus.

1 Connect corner to midpoint of opposite side, producing 30° angle **3** π **7** $\frac{\theta}{2\pi}\rightarrow$ area $\frac{1}{2}r^2\theta$
9 $d=1$, distance around hexagon < distance around circle **11** T; T; F; F
13 $\cos(2t+t)=\cos 2t\cos t-\sin 2t\sin t=4\cos^3 t-3\cos t$
15 $\frac{1}{2}\cos(s-t)+\frac{1}{2}\cos(s+t);\frac{1}{2}\cos(s-t)-\frac{1}{2}\cos(s+t)$ **17** $\cos\theta=\sec\theta=\pm 1$ at $\theta=n\pi$
19 Use $\cos(\frac{\pi}{2}-s-t)=\cos(\frac{\pi}{2}-s)\cos t+\sin(\frac{\pi}{2}-s)\sin t$ **23** $\theta=\frac{3\pi}{2}+$ multiple of 2π
25 $\theta=\frac{\pi}{4}+$ multiple of π **27** No θ **29** $\phi=\frac{\pi}{4}$ **31** $|OP|=a, |OQ|=b$

2 $\pi, 3\pi, -\frac{\pi}{4}$ radians equal **180°, 540°, −45°**. Also 60°, 90°, 270° equal $\frac{\pi}{3},\frac{\pi}{2},\frac{3\pi}{2}$ radians. The alias of 480° is **120°** and the alias of −1° is **359°**.
4 $\cos 2(\theta+\pi)$ is the same as $\cos(2\theta+2\pi)$ which is $\cos 2\theta$. Since $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$, this also has period π.
6 Notice the patterns in this table.
8 Straight distance $\sqrt{2}$; quarter-circle distance $\frac{\pi}{2}$; semicircle distance also $\frac{\pi}{2}$.
10 $d^2=(0-\frac{1}{2})^2+(1-\frac{\sqrt{3}}{2})^2=\frac{1}{4}+1-\sqrt{3}+\frac{3}{4}=2-\sqrt{3}$. Then $12d=6.21$. This is the distance around a twelve-sided figure that fits into the circle (curved distance is 2π.)
12 From the inside front cover or the addition formulas: $\sin(\pi-\theta)=\sin\theta, \cos(\pi-\theta)=-\cos\theta, \sin(\frac{\pi}{2}+\theta)=\cos\theta, \cos(\frac{\pi}{2}+\theta)=-\sin\theta$.
14 $\sin 3t=\sin(2t+t)=\sin 2t\cos t+\cos 2t\sin t$. This equals $(2\sin t\cos t)\cos t+(\cos^2 t-\sin^2 t)\sin t$ or $3\sin t-4\sin^3 t$.

16 $(\cos t + i\ \sin t)^2 = \mathbf{cos^2 t - sin^2 t + 2\, i\, sin\, t\, cos\, t}$. Then the double-angle formulas give $\cos 2t + i\ \sin 2t$.

18 A complete solution is not expected! Finding a point like $s = \pi/2, t = 3\pi/2$ is not bad.

20 Formula (9) is $\sin(s + t) = \sin s\ \cos t + \cos s\ \sin t$. **Replacing t by −t** gives formula (8) for $\sin(s - t)$. (Ask why this replacement is allowed. It is not easy for a student to explain.)

22 $\tan(s + t) = \frac{\sin(s+t)}{\cos(s+t)} = \frac{\sin s\ \cos t + \cos s \sin t}{\cos s\ \cos t - \sin s\ \sin t}$. To simplify, divide top and bottom by $\cos s$ and $\cos t$: $\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s\ \tan t}$.

24 $\sec\theta = -2$ when $\cos\theta = -\frac{1}{2}$, which happens first at $\theta = \mathbf{120°} = 2\pi/3$. Also at $\theta = \mathbf{240°} = 4\pi/3$. Then at all angles $\mathbf{2\pi/3 + 2\pi n}$ and $\mathbf{4\pi/3 + 2\pi n}$.

26 $\sin\theta = \theta$ at $\theta = \mathbf{0}$ **and never again**. Reason: The right side has slope 1 and the left side has slope $\cos\theta < 1$. (Draw graphs of $\sin\theta$ and θ. A solution with negative θ would give a solution for positive θ by reversing sign.)

28 $\tan\theta = 0$ when θ **is a multiple of** π. The ratio y/x is zero when $y = 0$, so the point on the circle in Figure 1.20 has to be on the x axis.

30 $A\ \sin(x + \phi)$ equals $A\ \sin\ x\cos\phi + A\ \cos x\ \sin\phi$. Matching with $a \sin x + b\cos x$ gives $a = A\cos\phi$ and $b = A\ \sin\phi$. Then $a^2 + b^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. Thus $\mathbf{A = \sqrt{a^2 + b^2}}$ and $\tan\phi = \frac{A\ \sin\phi}{A\ \cos\phi} = \frac{a}{b}$.

32 The distance squared from (0,0) to R is $(a + b\cos\theta)^2 + (b\ \sin\theta)^2$ which simplifies to $\mathbf{a^2 + 2ab\cos\theta + b^2}$. Notice the parallelogram law: (diagonal)^2 + (other diagonal)$^2 = 2a^2 + 2b^2$ which is (side)^2 + (next side)^2 + (third side)^2 + (fourth side)^2.

34 The amplitude and period of $2\ \sin\pi x$ are both **2**.

36 By Problem 30, $\sin x + \cos x$ equals $\mathbf{\sqrt{2}\,sin(x + \frac{\pi}{4})}$. The graph should show a sine function with maximum near $\sqrt{2}$ at $\mathbf{x = \frac{\pi}{4}}$.

38 The graph of $t\ \sin t$ oscillates **between ± 45° lines**. The graph of $\sin 4t\ \sin t$ oscillates **inside the graph of sin t**. See the graph on page 294, at the end of Section 7.2.