Table of Integrals

1 $\int u^n dx = \frac{u^{n+1}}{a(n+1)}$ except for $\int \frac{dx}{u} = \frac{\ln|u|}{a}$ All the integrals 1 - 17 involve $u = ax+b$

2 $\int xu^n\, dx = \frac{u^{n+2}}{a^2(n+2)} - \frac{bu^{n+1}}{a^2(n+1)}$ except for $\int \frac{x\,dx}{u} = \frac{x}{a} - \frac{b\ln|u|}{a^2}$ and $\int \frac{x\,dx}{u^2} = \frac{b}{a^2 u} + \frac{\ln|u|}{a^2}$

3 $\int \frac{x^2 dx}{u} = \frac{1}{a^3}(\frac{u^2}{2} - 2bu + b^2\ln|u|)$ **4** $\int \frac{x^2\,dx}{u^2} = \frac{1}{a^3}(u - 2b\ln|u| - \frac{b^2}{u})$ **5** $\int \frac{x^2 dx}{u^3} = \frac{1}{a^3}(\ln|u| + \frac{2b}{u} - \frac{b^2}{2u^2})$

6 $\int \frac{dx}{xu} = \frac{1}{b}\ln|\frac{x}{u}|$ **7** $\int \frac{dx}{x^2 u} = -\frac{1}{bx} + \frac{a}{b^2}\ln|\frac{u}{x}|$ **8** $\int \frac{dx}{xu^2} = \frac{1}{bu} - \frac{1}{b^2}\ln|\frac{u}{x}|$ **9** $\int \frac{dx}{x^2u^2} = -\frac{b+2ax}{b^2xu} + \frac{2a}{b^3}\ln|\frac{u}{x}|$

10 $\int \sqrt{u}\, dx = \frac{2}{3a}u^{3/2}$ **11** $\int x\sqrt{u}\, dx = \frac{2(3ax-2b)}{15a^2}u^{3/2}$ **12** $\int x^2\sqrt{u}\, dx = \frac{2(15a^2x^2-12abx+8b^2)}{105a^3}u^{3/2}$

13 $\int \frac{\sqrt{u}}{x}\, dx = 2\sqrt{u} + b\int \frac{dx}{x\sqrt{u}}$ **14** $\int \frac{x\,dx}{\sqrt{u}} = \frac{2(ax-2b)}{3a^2}\sqrt{u}$ **15** $\int \frac{x^2dx}{\sqrt{u}} = \frac{2(3a^2x^2-4abx+8b^2)}{15a^3}\sqrt{u}$

16 $\int \frac{dx}{x\sqrt{u}} = \frac{1}{\sqrt{b}}\ln|\frac{\sqrt{u}-\sqrt{b}}{\sqrt{u}+\sqrt{b}}|$ $(b>0)$ or $\frac{2}{\sqrt{-b}}\tan^{-1}\frac{\sqrt{u}}{\sqrt{-b}}$ $(b<0)$ **17** $\int \frac{\sqrt{u}}{x^2}dx = -\frac{\sqrt{u}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{u}}$

18 $\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\ln|\frac{cx+d}{ax+b}|$ **19** $\int \frac{x\,dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}(\frac{b}{a}\ln|ax+b| - \frac{d}{c}\ln|cx+d|)$

20 $\int \sqrt{x^2\pm a^2}\, dx = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln|x+\sqrt{x^2\pm a^2}|$ **21** $\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln|x+\sqrt{x^2\pm a^2}|$

22 $\int \frac{\sqrt{x^2+a^2}}{x}\, dx = \sqrt{x^2+a^2} - a\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right)$ **23** $\int \frac{\sqrt{x^2-a^2}}{x}\, dx = \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}$

24 $\int \frac{x^2\, dx}{\sqrt{x^2\pm a^2}} = \frac{x}{2}\sqrt{x^2\pm a^2} \mp \frac{a^2}{2}\ln|x+\sqrt{x^2\pm a^2}|$ **25** $\int \frac{\sqrt{x^2\pm a^2}}{x^2}\, dx = -\frac{\sqrt{x^2\pm a^2}}{x} + \ln|x+\sqrt{x^2\pm a^2}|$

26 $\int x^2\sqrt{x^2\pm a^2}\, dx = \frac{x}{8}(2x^2\pm a^2)\sqrt{x^2\pm a^2} - \frac{a^4}{8}\ln|x+\sqrt{x^2\pm a^2}|$ **27** $\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x}$

28 $\int (x^2\pm a^2)^{3/2}\, dx = \frac{x}{8}(2x^2\pm 5a^2)\sqrt{x^2\pm a^2} + \frac{3a^4}{8}\ln|x+\sqrt{x^2\pm a^2}|$ **29** $\int \frac{dx}{(x^2\pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2\pm a^2}}$

30 $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ **31** $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$ **32** $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}}$

33 $\int \frac{\sqrt{a^2-x^2}}{x}\, dx = \sqrt{a^2-x^2} - a\ln|\frac{a+\sqrt{a^2-x^2}}{x}|$ **34** $\int x^2\sqrt{a^2-x^2}\, dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a}$

35 $\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$ **36** $\int \frac{\sqrt{a^2-x^2}}{x^2}\, dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a}$ **37** $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$

38 $\int \frac{x^2\, dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ **39** $\int (a^2-x^2)^{3/2}dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\sin^{-1}\frac{x}{a}$

40 $\int \frac{dx}{b+c\sin ax} = \frac{-2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right]$, $b^2>c^2$ **41** $\int \frac{dx}{1+\sin ax} = -\frac{1}{a}\tan(\frac{\pi}{4}-\frac{ax}{2})$

42 $\int \frac{dx}{b+c\sin ax} = \frac{-1}{a\sqrt{c^2-b^2}}\ln\left|\frac{c+b\sin ax+\sqrt{c^2-b^2}\cos ax}{b+c\sin ax}\right|$, $b^2<c^2$ **43** $\int \frac{dx}{1-\sin ax} = \frac{1}{a}\tan(\frac{\pi}{4}+\frac{ax}{2})$

44 $\int \frac{dx}{b+c\cos ax} = \frac{2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\frac{ax}{2}\right]$, $b^2>c^2$ **45** $\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$

46 $\int \frac{dx}{b+c\cos ax} = \frac{1}{a\sqrt{c^2-b^2}}\ln|\frac{c+b\cos ax+\sqrt{c^2-b^2}\sin ax}{b+c\cos ax}|$, $b^2<c^2$ **47** $\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$

48 $\int \sin^{-1}ax\, dx = x\sin^{-1}ax + \frac{1}{a}\sqrt{1-a^2x^2}$ **49** $\int x^n\sin^{-1}ax\, dx = \frac{x^{n+1}}{n+1}\sin^{-1}ax - \frac{a}{n+1}\int \frac{x^{n+1}dx}{\sqrt{1-a^2x^2}}$

50 $\int \tan^{-1}ax\, dx = x\tan^{-1}ax - \frac{1}{2a}\ln(1+a^2x^2)$ **51** $\int x^n\tan^{-1}ax\, dx = \frac{x^{n+1}}{n+1}\tan^{-1}ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{1+a^2x^2}$

52 $\int e^{ax}dx = \frac{e^{ax}}{a}$ **53** $\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$ **54** $\int x^2e^{ax}\, dx = \frac{e^{ax}}{a^3}(a^2x^2-2ax+2)$ (b^{ax} is $e^{a(\ln b)x}$)

55 $\int \frac{dx}{x\ln ax} = \ln|\ln ax|$ Not elementary: $\int e^{x^2}\, dx$, $\int e^x\ln x\, dx$, $\int \frac{dx}{\ln x}$, $\int \frac{e^x}{x}\, dx$, $\int \frac{\sin x}{x}\, dx$, $\int \frac{\sin^{-1}x}{x}dx$

Exponentials and Logarithms

$y = b^x \leftrightarrow x = \log_b y \quad y = e^x \leftrightarrow x = \ln y$

$e = \lim(1+\frac{1}{n})^n = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828\cdots$

$e^x = \lim(1+\frac{x}{n})^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\ln y = \int_1^y \frac{dx}{x} \qquad \ln 1 = 0 \quad \ln e = 1$

$\ln xy = \ln x + \ln y \quad \ln x^n = n \ln x$

$\log_a y = (\log_a b)(\log_b y) \quad \log_a b = 1/\log_b a$

$e^{x+y} = e^x e^y \quad b^x = e^{x \ln b} \quad e^{\ln y} = y$

Equations and Their Solutions

$y' = cy$	$y_0 e^{ct}$
$y' = cy + s$	$y_0 e^{ct} + \frac{s}{c}(e^{ct} - 1)$
$y' = cy - by^2$	$\frac{c}{b + de^{-ct}} \quad d = \frac{c - by_0}{y_0}$
$y'' = -\lambda^2 y$	$\cos \lambda t$ and $\sin \lambda t$
$my'' + dy' + ky = 0$	$e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ or $te^{\lambda_1 t}$
$y_{n+1} = ay_n$	$a^n y_0$
$y_{n+1} = ay_n + s$	$a^n y_0 + s\frac{a^n - 1}{a - 1}$

Vectors and Determinants

$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A} = a_1^2 + a_2^2 + a_3^2$ (length squared)

$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{A}||\mathbf{B}|\cos\theta$

$|\mathbf{A}\cdot\mathbf{B}| \le |\mathbf{A}||\mathbf{B}|$ (Schwarz inequality: $|\cos\theta| \le 1$)

$|\mathbf{A}+\mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$ (triangle inequality)

$|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}||\sin\theta|$ (cross product)

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{matrix} \mathbf{i}(a_2b_3 - a_3b_2) \\ +\mathbf{j}(a_3b_1 - a_1b_3) \\ +\mathbf{k}(a_1b_2 - a_2b_1) \end{matrix}$$

Right hand rule $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$

Parallelogram area $= |a_1b_2 - a_2b_1| = |\text{Det}|$

Triangle area $= \frac{1}{2}|a_1b_2 - a_2b_1| = \frac{1}{2}|\text{Det}|$

Box volume $= |\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})| = |\text{Determinant}|$

Matrices and Inverses

$Ax =$ combination of columns $= b$

Solution $x = A^{-1}b$ if $A^{-1}A = I$

Least squares $A^TA\overline{x} = A^Tb$

$Ax = \lambda x \quad$ (λ is an eigenvalue)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1}$, $(AB)^T = B^TA^T$

$$\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}^{-1} = \frac{1}{D}\begin{bmatrix} \mathbf{b}\times\mathbf{c} \\ \mathbf{c}\times\mathbf{a} \\ \mathbf{a}\times\mathbf{b} \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{matrix} +\,a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \\ -\,a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \end{matrix}$$

	SI Units	Symbols
length	meter	m
mass	kilogram	kg
time	second	s
current	ampere	A
frequency	hertz	Hz ~ 1/s
force	newton	N ~ kg·m/s^2
pressure	pascal	Pa ~ N/m^2
energy, work	joule	J ~ N·m
power	watt	W ~ J/s
charge	coulomb	C ~ A·s
temperature	kelvin	K

Speed of light $c = 2.9979 \times 10^8$ m/s

Gravity $G = 6.6720 \times 10^{-11}$ Nm^2/kg^2

From	To	Multiply by
degrees	radians	.01745
calories	joules	4.1868
BTU	joules	1055.1
foot-pounds	joules	1.3558
feet	meters	.3048
miles	km	1.609
feet/sec	km/hr	1.0973
pounds	kg	.45359
ounces	kg	.02835
gallons	liters	3.785
horsepower	watts	745.7

Radius at Equator $R = 6378$ km $= 3964$ miles

Acceleration $g = 9.8067$ m/s^2 $= 32.174$ ft/s^2

Sums and Infinite Series

$1+x+\cdots+x^{n-1}=\frac{1-x^n}{1-x}$

$1+nx+\frac{n(n-1)}{2!}x^2+\cdots+x^n=(1+x)^n$

$1+2+\cdots+n=\frac{1}{2}n(n+1)\approx\frac{n^2}{2}$

$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}\approx\frac{n^3}{3}$

$1+\frac{1}{2}+\cdots+\frac{1}{n}\approx\ln n\to\infty$ (harmonic)

$1-\frac{1}{2}+\frac{1}{3}-\cdots=\ln 2$ (alternating)

$1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{\pi}{4}\quad\sum\frac{1}{n^2}=\frac{\pi^2}{6}\quad\sum\frac{1}{n^4}=\frac{\pi^4}{90}$

$\frac{1}{1-x}=1+x+x^2+\cdots$ (geometric: $|x|<1$)

$\frac{1}{(1-x)^2}=1+2x+3x^2+\cdots=\frac{d}{dx}\left(\frac{1}{1-x}\right)$

$\frac{1}{1+x}=1-x+x^2-\cdots$ (geometric for $-x$)

$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots=\int\frac{dx}{1+x}$

$\sin x=x-x^3/6+x^5/120-\cdots$ (all x)

$\cos x=1-x^2/2+x^4/24-\cdots$ (all x)

$e^x=1+x+\frac{x^2}{2!}+\cdots(e=1+1+\frac{1}{2!}+\cdots)$

$e^{ix}=\cos x+i\sin x$ (Euler's formula)

$\cosh x = \frac{1}{2}(e^x+e^{-x}) = 1+\frac{x^2}{2!}+\cdots$

$\sinh x = \frac{1}{2}(e^x-e^{-x}) = x+\frac{x^3}{3!}+\cdots$

$(\cos\theta+i\ \sin\theta)^n=\cos n\theta+i\sin n\theta$

$f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+\cdots$ (Taylor)

$f(x,y)=f+xf_x+yf_y+\frac{x^2}{2!}f_{xx}+xyf_{xy}+\cdots$

Polar and Spherical

$x=r\cos\theta$ and $y=r\sin\theta$

$r=\sqrt{x^2+y^2}$ and $\tan\theta=y/x$

$x+iy=r(\cos\theta+i\ \sin\theta)=re^{i\theta}$

Area $\int\frac{1}{2}r^2d\theta$ Length $\int\sqrt{r_\theta^2+r^2}d\theta$

$x=\rho\sin\phi\cos\theta, y=\rho\sin\phi\sin\theta, z=\rho\cos\phi$

Area $dA=dx\ dy=r\ dr\ d\theta=J\ du\ dv$

Volume $r\ dr\ d\theta\ dz=\rho^2\sin\phi\ d\rho\ d\phi\ d\theta$

Stretching factor $J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}x_u & x_v\\ y_u & y_v\end{vmatrix}$

Area - Volume - Length - Mass - Moment

Circle πr^2 Ellipse πab Wedge of circle $r^2\theta/2$

Cylinder side $2\pi rh$ Volume πr^2h Shell $dV=2\pi rh\ dr$

Sphere surface $4\pi r^2$ Volume $\frac{4}{3}\pi r^3$ Shell $dV=4\pi r^2dr$

Cone or pyramid Volume $\frac{1}{3}$ (base area) (height)

Length of curve $\int ds=\int\sqrt{1+(dy/dx)^2}\ dx$

Area between curves $\int(v(x)-w(x))dx$

Surface area of revolution $\int 2\pi r\ ds(r=x$ or $r=y)$

Volume of revolution: Slices $\int\pi y^2dx$ Shells $\int 2\pi xh\ dx$

Area of surface $z(x,y):\int\int\sqrt{1+z_x^2+z_y^2}\ dx\ dy$

Mass $M=\int\int\rho\ dA$ Moment $M_y=\int\int\rho x\ dA$

$\overline{x}=M_y/M,\overline{y}=M_x/M$ Moment of Inertia $I_y=\int\int\rho x^2dA$

Work $W=\int_a^b F(x)dx=V(b)-V(a)$ Force $F=dV/dx$

Partial Derivatives of $z=f(x,y)$

Tangent plane $z-z_0=(\frac{\partial f}{\partial x})(x-x_0)+(\frac{\partial f}{\partial y})(y-y_0)$

Approximation $\Delta z\approx(\frac{\partial f}{\partial x})\Delta x+(\frac{\partial f}{\partial y})\Delta y$

Normal $\mathbf{N}=(f_x,f_y,-1)$ or (F_x,F_y,F_z)

Gradient $\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}$

Directional derivative: $D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}=f_xu_1+f_yu_2$

Chain rule: $\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}$

Vector field $\mathbf{F}(x,y,z)=M\ \mathbf{i}+N\ \mathbf{j}+P\ \mathbf{k}$

Work $\int\mathbf{F}\cdot d\mathbf{R}$ Flux $\int M\ dy-Ndx$

Divergence of $\mathbf{F}=\nabla\cdot\mathbf{F}=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}$

Curl of $\mathbf{F}=\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ M & N & P\end{vmatrix}$

Conservative $\mathbf{F}=\nabla f=$ gradient of f if curl $\mathbf{F}=\mathbf{O}$

Green's Theorem $\oint M\ dx+N\ dy=\int\int(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y})\ dx\ dy$

Divergence Theorem $\int\int\mathbf{F}\cdot\mathbf{n}\ dS=\int\int\int$ div $\mathbf{F}\ dV$

Stokes' Theorem $\oint\mathbf{F}\cdot d\mathbf{R}=\int\int$ (curl $\mathbf{F})\cdot\mathbf{n}\ dS$

An additional table of integrals is included just after the index.