Recitation 6

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1 Combinatorics

We consider the 3-point interaction $\mathcal{L}_I = \frac{\lambda}{3!}\phi^3$, and compute to second order in λ the object:

$$G_2 = \langle\Omega|\mathrm{T}\phi(x_1)\phi(x_2)|\Omega\rangle = \frac{\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)e^{\frac{i\lambda}{3!}\int d^4z\phi^3(z)}|0\rangle}{\langle 0|\mathrm{T}e^{\frac{i\lambda}{3!}\int d^4z\phi^3(z)}|0\rangle} = \frac{N}{D}$$

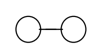
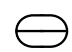
We will compute this object using 3 different approaches to simplify the combinatorics, each simpler and more useful than the last.

1.1 Wick Contractions

We start with the denominator.

$$\begin{aligned}\langle 0|\mathrm{T}e^{\frac{i\lambda}{3!}\int d^4z\phi^3(z)}|0\rangle &= \sum_{s=0}^{2}\frac{(i\lambda/3!)^s}{s!}\int d^4z_1\cdots d^4z_s\langle 0|\phi^3(z_1)\cdots\phi^3(z_s)|0\rangle + \mathcal{O}(\lambda^3)\\ &= 1 + i\lambda\int d^4z\langle 0|\phi^3(z)|0\rangle - \frac{\lambda^2}{2(3!)^2}\int d^4z_1d^4z_2\langle 0|\phi^3(z_1)\phi^3(z_2)|0\rangle\end{aligned}$$

The order λ^1 term vanishes, since it is a correlator of an odd number of fields, so there are no ways to Wick-pair them. It remains to evaluate the object $\langle 0|\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)|0\rangle$. There are two distinct ways of Wick contracting, which we summarize in a table:

Contraction Type	Ways to Contract	Value (per contraction)	Diagram
$\langle\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)\rangle$	3×3	$G_F(z_1,z_1)G_F(z_1,z_2)G_F(z_2,z_2)$	
$\langle\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)\rangle$	$3\times 2\times 1$	$G_F(z_1,z_2)^3$	

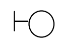
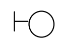
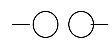
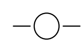
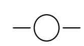
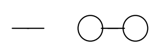
Putting everything together, we have

$$\begin{aligned}D &= 1 - \lambda^2\int d^4z_1d^4z_2\left[\frac{1}{8}G_F(0,0)^2G_F(z_1,z_2) + \frac{1}{12}G_F(z_1,z_2)^3\right] + \mathcal{O}(\lambda^3)\\ \frac{1}{D} &= 1 + \lambda^2\int d^4z_1d^4z_2\left[\frac{1}{8}G_F(0,0)^2G_F(z_1,z_2) + \frac{1}{12}G_F(z_1,z_2)^3\right] + \mathcal{O}(\lambda^3)\end{aligned}$$

Now the numerator.

$$
\begin{aligned}
\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)e^{\frac{i\lambda}{3!}\int d^4z\phi^3(z)}|0\rangle &= \sum_{s=0}^{2}\frac{(i\lambda/3!)^s}{s!}\int d^4z_1\cdots d^4z_s\langle 0|\phi(x_1)\phi(x_2)\phi^3(z_1)\cdots\phi^3(z_s)|0\rangle + \mathcal{O}(\lambda^3)\\
&= \langle 0|\phi(x_1)\phi(x_2)|0\rangle + i\lambda\int d^4z\langle 0|\phi(x_1)\phi(x_2)\phi^3(z)|0\rangle\\
&\quad - \frac{\lambda^2}{2(3!)^2}\int d^4z_1d^4z_2\langle 0|\phi(x_1)\phi(x_2)\phi^3(z_1)\phi^3(z_2)|0\rangle
\end{aligned}
$$

The first term is the result from the free theory, in the absence of interactions. The linear term in λ is again odd in ϕ's, so vanishes. We need to compute the quadratic term. There are several ways of Wick contracting, given below.

Contraction Type	# Ways	Value (per contraction)	Diagram	Notes
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	$3\times 2\times 3$	$G^F_{x_1,z_1}G^F_{x_2,z_1}G^F_{z_1,z_2}G^F_{z_2,z_2}$		
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	$3\times 2\times 3$	$G^F_{x_1,z_2}G^F_{x_2,z_2}G^F_{z_1,z_1}G^F_{z_1,z_2}$		above with $z_1\leftrightarrow z_2$
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	3×3	$G^F_{x_1,z_1}G^F_{x_2,z_2}G^F_{z_1,z_1}G^F_{z_2,z_2}$		
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	3×3	$G^F_{x_1,z_2}G^F_{x_2,z_1}G^F_{z_1,z_1}G^F_{z_2,z_2}$		above with $z_1\leftrightarrow z_2$
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	$3\times 3\times 2$	$G^F_{x_1,z_1}G^F_{x_2,z_2}G^F_{z_1,z_2}G^F_{z_1,z_2}$		
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	$3\times 3\times 2$	$G^F_{x_1,z_2}G^F_{x_2,z_1}G^F_{z_1,z_2}G^F_{z_1,z_2}$		above with $z_1\leftrightarrow z_2$
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	3×3	$G^F_{x_1,x_2}G^F_{z_1,z_1}G^F_{z_1,z_2}G^F_{z_2,z_2}$		vacuum
$\langle\phi_{x_1}\phi_{x_2}\phi_{z_1}\phi_{z_1}\phi_{z_1}\phi_{z_2}\phi_{z_2}\phi_{z_2}\rangle$	3×2	$G^F_{x_1,x_2}G^F_{z_1,z_2}G^F_{z_1,z_2}G^F_{z_1,z_2}$		vacuum

Putting everything together, we have

$$
\begin{aligned}
\frac{N}{D} &= \frac{G_F(x_1,x_2)(1+\text{vac.})+N_{\text{no vac.}}}{1+\text{vac.}} + \mathcal{O}(\lambda^3) = G_F(x_1,x_2) + N_{\text{no vac.}} + \mathcal{O}(\lambda^3)\\
&= G_F(x_1,x_2) - \frac{\lambda^2}{2}\int d^4z_1d^4z_2[G_{x_1,z_1}G_{x_2,z_1}G_{z_1,z_2}G_{0,0} + \frac{1}{2}G_{x_1,z_1}G_{x_2,z_2}G_{0,0}^2 +\\
&\qquad + G_{x_1,z_1}G_{x_2,z_2}G_{z_1,z_2}G_{z_1,z_2}]
\end{aligned}
$$

1.2 Feynman Rules in Position Space

The Feynman rules give a prescription to evaluate the Gell-Mann Low formula, which takes less thinking than Wick contractions, although the combinatorics (read: symmetry factors) can be more confusing. The idea is to write down a set of diagrams corresponding to the terms in the Taylor expansion of e^{iS_I}. We can compute each diagram using the Feynman rules, and adding a combinatorial factor.

The procedure is as follows.

1. Draw the external and internal vertices to desired order in perturbation theory.
    - An external vertex has 1 leg, and correspond to the position (or momenta) of fields in the correlator. For a given correlator, the number of external vertices is always the same.
    - An internal vertex corresponding to $S_I = \lambda\phi^k/k!$ has k legs. Each external vertex corresponds to a position (or momentum) z which is integrated over, coming from a power of $S_I = \int d^dz(\cdots)$ when Taylor expanded. To compute up to λ^r in perturbation theory, one must consider all diagrams with $\leq r$ internal vertices.
2. Connect the vertices using the legs in all (topologically distinct) ways to get all of the diagrams. Drop diagrams which have vacuum bubbles (i.e. a disconnected component with no external legs attached). These are cancelled out by the denominator in the Gell-Mann Low formula (see: your problem set.)
3. Feynman rules. For each diagram, write its contribution using the following 'Feynman rules'.
    - Each external vertex contributes as 1. Sometimes you may see this written with a factor $e^{-ix\cdot p}$: this is just a Fourier factor to convert to momentum space.
    - Each internal vertex corresponding to a term $i\lambda\phi^k/k! \subset iS_I$ contributes as $i\lambda$.
    - Each edge between vertices corresponds to a Wick contraction between the fields at those points, hence contributes $G_F(x-y)$.
4. Symmetry factors. This is the most confusing step.
   Note that the Feynman rules in (3) don't include any $k!$ or $r!$ (perturbation order) combinatorial factors. This is because we have assumed naïve combinatorics, amounting to 2 statements:
    (a) There are $k!$ ways to permute the legs of each vertex, each of which prescribe a way to 'connect' the vertex to the rest of the diagram via Wick contractions. If each permutation leads to a distinct contraction, we pick up a $k!$ factor which cancels out with the $1/k!$ factor in associated to the vertex in S_I. Hence, we can ignore both.
    (b) Working to order r in perturbation theory, there are $r!$ ways to permute the different internal vertices. If each permutation leads to a distinct set of contractions, we pick up an $r!$ factor which cancels out with the $1/r!$ in the Taylor expansion of the exponential. Hence, we can ignore both.

   Corrected combinatorics: there are certain situations where this naïve counting breaks down. In particular, some permutation of vertex legs or internal vertices lead to the same Wick contraction, so the naïve prescription is overcounting. To correct this we divide by a 'symmetry factor', which should be assigned (multiplicatively) in the following cases:

    (a) Propagators starting and ending at the same point: $\times\frac{1}{2}$
    Since these are Wick contracted, permuting them leads to the same Wick contraction.

(b) j propagators connecting 2 internal vertices: $\times\frac{1}{j!}$
Naïvely we pick up a factor of $j!$ from permuting these legs of each vertex, so $j!^2$ in total. However, there are only $j!$ ways to Wick contract, hence we overcount by $j!$.

(c) j internal vertices which are interchangeable in how they connect to the diagram: $\times\frac{1}{j!}$
Permuting these vertices lead to identical Wick contractions, so we naïvely overcount by $j!$

Let us use these to compute the example from above. We have the following diagrams and their contributions. Note that $G_F(z,z) = G_F(0,0)$ by translation symmetry.

Diagram	Symm. Factor	Value
$-\bigcirc-$	$\frac{1}{2}$	$\frac{1}{2}(i\lambda)^2 \int d^4z_1 d^4z_2 G_F(x_1,z_1)G_F^2(z_1,z_2)G_F(x_2,z_2)$
$\vdash\!\bigcirc$	$\frac{1}{2}$	$\frac{1}{2}(i\lambda)^2 G_F(0,0) \int d^4z_1 d^4z_2 G_F(x_1,z_1)G_F(x_2,z_1)G_F(z_1,z_2)$
$-\bigcirc\ \bigcirc-$	$\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{4}(i\lambda)^2 G_F^2(0,0) \int d^4z_1 G_F(x_1,z_1) \int d^4z_2 G_F(x_2,z_2)$
$--$ (vac.)	$-$	N/A

Together with the 0th order piece, we have our result from before, using the shorthand $G_{x,y} := G_F(x,y)$.

$$G_2 = G_{x_1,x_2} - \frac{\lambda^2}{2}\int d^4z_1 d^4z_2 [G_{x_1,z_1}G_{x_2,z_1}G_{z_1,z_2}G_{0,0} + \frac{1}{2}G_{x_1,z_1}G_{x_2,z_2}G_{0,0}^2 + G_{x_1,z_1}G_{x_2,z_2}G_{z_1,z_2}G_{z_1,z_2}]$$

1.3 Feynman Rules in Momentum Space

These are very similar to the Feynman rules in position space. However, the diagrams are simpler to evaluate because we can make use of momentum conservation and the explicit form of the propagator in momentum space to simplify our expression.

The procedure is as follows.

1. Draw the external and internal vertices to desired order in perturbation theory. (same)
2. Draw down all topologically distinct diagrams without vacuum bubbles. (same)
3. Feynman rules: these are now different.
   - Each external vertex contributes as 1. Sometimes you may see this written with a factor $e^{ix\cdot p}$: this is just a Fourier factor to convert to position space.
   - Each internal vertex contributes $i\lambda\delta^{(4)}(\sum p_{\text{in}} - \sum p_{\text{out}})$.
     - Note: we can see how this comes from taking the position space Feynman rules: we have a Fourier phase for each fields at an internal vertex point x, totalling to $e^{-ix(\sum iq_i)}$. Integrating over the internal vertex location x produces the desired δ-function.

   We account for the δ-function by imposing momentum conservation at each vertex. This fixes all internal momenta, aside from those running in a loop, which are still integrated over. This includes the total momentum conservation $(2\pi)^4\delta^{(4)}(\sum p_{\text{in}} - \sum p_{\text{out}})$
   - Each edge (Wick contraction) contributes $G_F(p) = \dfrac{-i}{p^2+m^2-i\epsilon}$
4. Symmetry factors: these are the same as in position space. (same)

Let us use these to compute the example from above. We have the following diagrams and their contributions:

Diagram	Symm.	Value
−◯−	$\frac{1}{2}$	$\frac{1}{2}(i\lambda)^2(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\times \quad \int\frac{\text{đ}^4k(-i)^2}{(k^2+m^2-i\epsilon)((p-k)^2+m^2-i\epsilon)}$
⊢◯	$\frac{1}{2}$	$\frac{1}{2}(i\lambda)^2(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\times \quad \frac{-i}{(p_1-p_2)^2+m^2-i\epsilon}\int\frac{\text{đ}^4k(-i)}{k^2+m^2-i\epsilon}$
−◯ ◯−	$\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{4}(i\lambda)^2(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{-i}{p_1^2+m^2-i\epsilon}\frac{-i}{p_2^2+m^2-i\epsilon}\times \quad \left(\int\frac{\text{đ}^4k(-i)}{k^2+m^2-i\epsilon}\right)^2$
−− (vac.)	−	N/A

Summing the contributions, we get

$$G_2(p_1,p_2) = -\frac{\lambda^2}{2}(2\pi)^4\delta^{(4)}(p_1-p_2)\frac{1}{p_1^2+m^2-i\epsilon}\frac{1}{p_2^2+m^2-i\epsilon}$$
$$\left[\int\frac{\text{đ}^4k}{(k^2+m^2-i\epsilon)((p-k)^2+m^2-i\epsilon)}+\frac{1}{(p_1-p_2)^2+m^2-i\epsilon}\int\frac{\text{đ}^4k}{k^2+m^2-i\epsilon}+\frac{1}{2}\left(\int\frac{\text{đ}^4k}{k^2+m^2-i\epsilon}\right)^2\right]$$