8.323 Problem Set 2 Solutions

February 21, 2023

Question 1: A Problem with Relativistic Quantum Mechanics (20 points)

The Schrödfinger equation for a free non-relativistic particle is:

$$i\partial_t\psi(\vec{x},t) = -\frac{1}{2m}\nabla^2\psi(\vec{x},t)$$

The generalization of the above equation to a free relativistic particle is the so-called Klein-Gordon equation

$$\partial_t^2\psi(\vec{x},t) - \nabla^2\psi(\vec{x},t) + m^2\psi(\vec{x},t) = 0$$

We emphasize that in both these equations, $\psi(\vec{x},t)$ is interpreted as a wave function for the dynamical variable $\vec{x}(t)$, rather than a dynamical field.

(a) As a reminder, derive from the Schrödinger equation the continuity equation for the probability

$$\partial_t\rho + \nabla\cdot\vec{J} = 0$$

where

$$\rho = |\psi|^2, \qquad \vec{J} = -\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

We compute:

$$\begin{aligned}\partial_t\rho = \psi\partial_t\psi^* + \psi^*\partial_t\psi &= -\frac{i}{2m}\left(\psi\vec{\nabla}^2\psi^* - \psi^*\vec{\nabla}^2\psi\right)\\ &= \frac{i}{2m}\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = -\vec{\nabla}\cdot\vec{J}\end{aligned}$$

where we use the Schrödinger equation in the second equality.

(b) Suppose $\psi(\vec{x},t)$ has the plane wave form, i.e.

$$\psi(\vec{x},t) \propto e^{i\vec{k}\cdot\vec{x}}$$

for some real vector $\vec{k}$. Find the solutions to the Klein-Gordon equation above.

We substitute the ansatz $\psi(\mathbf{x},t) = e^{i\mathbf{k}\cdot\mathbf{x}}\phi(t)$ into the Klein-Gordon equation to get an equation for $\phi(t)$:

$$\partial_t^2\phi + (\mathbf{k}^2 + m^2)\phi = 0$$

This has plane-wave solutions of positive and negative frequencies,

$$\phi(t) = Ae^{-i\omega_{\mathbf{k}}t} + Be^{i\omega_{\mathbf{k}}t}, \qquad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$$

Hence, the Klein-Gordon equation has solutions

$$\psi(\vec{x},t) = Ae^{i(-\omega_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})} + Be^{i(\omega_{\mathbf{k}}t+\mathbf{k}\cdot\mathbf{x})}$$

(c) Show that the Klein-Gordon equation also leads to a continuity equation, with ρ and $\vec{J}$ now given by

$$\rho = \frac{i}{2m}(\psi^*\partial_t\psi - \psi\partial_t\psi^*). \qquad \vec{J} = -\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

In the same way as in part (a), we compute:

$$\partial_t\rho = \frac{i}{2m}\left(\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^*\right) = \frac{i}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = -\vec{\nabla}\cdot\vec{J}$$

where we use the Klein-Gordon equation in the second equality.

(d) Argue that this ρ cannot be interpreted as a probability density.
We write

$$\rho = \frac{i}{2m}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right) = \frac{1}{m}\mathrm{Im}(\psi\partial_t\psi^*)$$

Any proper probability density must be positive definite, i.e. $\rho \geq 0$. This is not the case here. For instance, for the plane wave solution $A = 0$, B from (b) we compute

$$\rho = \mathrm{Im}\left(Be^{i\omega_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}}(-i\omega_{\mathbf{k}})Be^{-i\omega_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}}\right) = -B^2\omega_{\mathbf{k}} < 0$$

Since $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2} > 0$, the existence of negative-frequency solutions means that ρ cannot be positive definite, and cannot be interpreted as a probability density.

Question 2: Commutation relations of creation and annihilation operators (20 points)
For the real scalar field theory discussed in lecture,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

we showed that the time-evolution of the quantum operator $\phi(\mathbf{x}, t)$ is given by

$$\phi(\mathbf{x}, t) = \int \not{d}^2k \frac{1}{\sqrt{2\omega_\mathbf{k}}}\left(a_\mathbf{k}u_\mathbf{k}(\mathbf{x}, t) + a^\dagger_\mathbf{k}u^*_\mathbf{k}(\mathbf{x}, t)\right)$$

where

$$\omega_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}, \qquad u_\mathbf{k} = e^{-i\omega_\mathbf{k}t+i\mathbf{k}\cdot\mathbf{x}}$$

We use $\pi(\mathbf{x}, t)$ to denote the momentum density conjugate to ϕ. The canonical commutation relations among ϕ and π are

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = 0, \qquad [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

(a) Show that it is enough to impose the canonical commutation relations at $t = 0$. That is, once we impose them at $t = 0$, then the relations at general t are automatically satisfied.
Note: this statement in fact applies not only to $V(\phi) = \frac{1}{2}m^2\phi^2$, but any potential $V(\phi)$.

In the Heisenberg picture we have:

$$[A(\mathbf{x}, t), B(\mathbf{x}', t)] = [e^{iHt}A(\mathbf{x}, 0)e^{-iHt}, e^{iHt}B(\mathbf{x}', 0)e^{-iHt}] = e^{iHt}[A(\mathbf{x}, 0), B(\mathbf{x}', 0)]e^{-iHt}$$

Now let us impose the canonical commutation relations at $t = 0$. Then, it follows that

$$[\phi(\mathbf{x}, t), \phi(\mathbf{x}', t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = e^{iHt}0e^{-iHt} = 0$$
$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = e^{iHt}i\delta^{(3)}(\mathbf{x} - \mathbf{x}')e^{-iHt} = i\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

These are again precisely the canonical commutation relations, now at generic t.

(b) Express $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$ in terms of $\phi(\mathbf{k})$ and $\pi(\mathbf{k})$, where $\phi(\mathbf{k})$ and $\pi(\mathbf{k})$ are Fourier transforms of $\phi(\mathbf{x}, t = 0)$ and $\pi(\mathbf{x}, t = 0)$, e.g.

$$\phi(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}, t = 0)$$

We start with the mode expansions for $\phi(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$

$$\phi(\mathbf{x}, t) = \int \not{d}^3k \frac{1}{\sqrt{2\omega_\mathbf{k}}}\left(a_\mathbf{k}e^{-i\omega_\mathbf{k}t+i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_\mathbf{k}e^{i\omega_\mathbf{k}t-i\mathbf{k}\cdot\mathbf{x}}\right)$$
$$\pi(\mathbf{x}, t) = -i\int \not{d}^3k \sqrt{\frac{\omega_\mathbf{k}}{2}}\left(a_\mathbf{k}e^{-i\omega_\mathbf{k}t+i\mathbf{k}\cdot\mathbf{x}} - a^\dagger_\mathbf{k}e^{i\omega_\mathbf{k}t-i\mathbf{k}\cdot\mathbf{x}}\right)$$

This is almost of the form of a Fourier transform, and by changing variables of one of the terms from $\mathbf{k} \to -\mathbf{k}$ we have

$$\phi(\mathbf{k}, t) = \frac{1}{\sqrt{2\omega_\mathbf{k}}}\left(a_\mathbf{k}e^{-i\omega_\mathbf{k}t} + a^\dagger_{-\mathbf{k}}e^{i\omega_\mathbf{k}t}\right)$$
$$\pi(\mathbf{k}, t) = -i\sqrt{\frac{\omega_\mathbf{k}}{2}}\left(a_\mathbf{k}e^{-i\omega_\mathbf{k}t} - a^\dagger_{-\mathbf{k}}e^{i\omega_\mathbf{k}t}\right)$$

Now, observe that the equations are decoupled in $\mathbf{k}$. We can take $t=0$ and solve this as a regular system of equations for $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$.

$$\begin{aligned} a_{\mathbf{k}} &= \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\phi(\mathbf{k}) + i\sqrt{\frac{1}{2\omega_{\mathbf{k}}}}\pi(\mathbf{k}) \\ a_{\mathbf{k}}^{\dagger} &= \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\phi(-\mathbf{k}) - i\sqrt{\frac{1}{2\omega_{\mathbf{k}}}}\pi(-\mathbf{k}) \end{aligned} \tag{1}$$

(c) Using the expressions derived in part (b), deduce the commutation relations

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}], \qquad [a_{\mathbf{k}}^{\dagger}, a_{\mathbf{k}'}^{\dagger}], \qquad [a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}]$$

from the commutation relations above at $t=0$.

It is useful to take the Fourier transform $\mathcal{F}$ (from position to momentum space) of the $t=0$ canonical commutation relations:

$$\begin{aligned} [\phi(\mathbf{k}), \phi(\mathbf{k}')] &= [\pi(\mathbf{k}), \pi(\mathbf{k}')] = \mathcal{F}_{\mathbf{x}\to\mathbf{k}} \circ \mathcal{F}_{\mathbf{x}'\to\mathbf{k}'}(0) = 0 \\ [\phi(\mathbf{k}), \pi(\mathbf{k}')] &= \mathcal{F}_{\mathbf{x}\to\mathbf{k}} \circ \mathcal{F}_{\mathbf{x}'\to\mathbf{k}'}(i\delta^{(3)}(\mathbf{x}-\mathbf{x}')) = i\int d^3\mathbf{x} d^3\mathbf{x}' e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}'}\delta^{(3)}(\mathbf{x}-\mathbf{x}') \\ &= i\int d^3\mathbf{x} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} = i(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}') \end{aligned}$$

Now we compute commutators of creation and annihilation operators using the results in (b)

$$\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{k}'}] &= +\frac{i}{2}\left([\phi(\mathbf{k}), \pi(\mathbf{k}')] + [\pi(\mathbf{k}), \phi(\mathbf{k}')]\right) &&= -\frac{1}{2}(2\pi)^3(\delta^{(3)}(\mathbf{k}+\mathbf{k}') - \delta^{(3)}(\mathbf{k}'+\mathbf{k})) &&= 0 \\ [a_{\mathbf{k}}^{\dagger}, a_{\mathbf{k}'}^{\dagger}] &= -\frac{i}{2}\left([\phi(-\mathbf{k}), \pi(-\mathbf{k}')] + [\pi(-\mathbf{k}), \phi(-\mathbf{k}')]\right) &&= -\frac{1}{2}(2\pi)^3(\delta^{(3)}(-\mathbf{k}-\mathbf{k}') - \delta^{(3)}(-\mathbf{k}'-\mathbf{k})) &&= 0 \\ [a_{\mathbf{k}}, a_{\mathbf{k}'}^{\dagger}] &= +\frac{i}{2}\left(-[\phi(\mathbf{k}), \pi(-\mathbf{k}')] + [\pi(\mathbf{k}), \phi(-\mathbf{k}')]\right) &&= -\frac{1}{2}(2\pi)^3(-\delta^{(3)}(\mathbf{k}-\mathbf{k}') - \delta^{(3)}(\mathbf{k}'-\mathbf{k})) \\ &= (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \end{aligned} \tag{2}$$

Question 3: Noether charges in terms of creation and annihilation operators (20 points)
In problem set 1, we obtained the conserved charges associated with spacetime translational symmetries for a complex scalar field theory. The results there can be easily converted to the corresponding expressions for a real scalar field theory.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

(a) Express the Hamiltonian H of this theory in terms of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$.
From problem set 1, we quote

$$H = \frac{1}{2}\int d^3x(\pi^2 + (\nabla\phi)^2 + m^2\phi^2)$$

It is convenient to first convert this expression into momentum space, before using the decomposition into creation and annihilation operators. We use the identity:

$$\begin{aligned}\int d^3\mathbf{x} f(\mathbf{x})g(\mathbf{x}) &= \int d^3\mathbf{x}d^3\mathbf{k}d^3\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} f(\mathbf{k})g(\mathbf{k}') \\ &= \frac{1}{(2\pi)^3}\int d^3\mathbf{k}d^3\mathbf{k}'\delta^{(3)}(\mathbf{k}+\mathbf{k}')f(\mathbf{k})g(\mathbf{k}') = \int đ^3\mathbf{k} f(\mathbf{k})g(-\mathbf{k}) \end{aligned} \tag{3}$$

More generally if there are derivatives acting on f or g, each derivative acting on f drags down a factor of $i\mathbf{k}$, while each derivative actin on g drags down a factor of $i\mathbf{k}'$, which becomes $-i\mathbf{k}$ after performing the $d^3\mathbf{x}$ integral. We further use the shorthand $đx = dx/2\pi$ ($đ$ is to d as $\hbar$ is to h).

Hence, we can now write

$$\begin{aligned} H &= \frac{1}{2}\int đ^3\mathbf{k}\left(\pi(\mathbf{k},t)\pi(-\mathbf{k},t) + (\mathbf{k}^2+m^2)\phi(\mathbf{k},t)\phi(-\mathbf{k},t)\right) \\ &= \frac{1}{2}\int đ^3\mathbf{k}\left(\pi(\mathbf{k},t)\pi(-\mathbf{k},t) + \omega_{\mathbf{k}}^2\phi(\mathbf{k},t)\phi(-\mathbf{k},t)\right) \\ &= \frac{1}{2}\int đ^3\mathbf{k}\frac{\omega_{\mathbf{k}}}{2}\left(a_{\mathbf{k}}(t)a_{\mathbf{k}}(t)^\dagger + a_{\mathbf{k}}(t)^\dagger a_{\mathbf{k}}(t)\right) = \int đ^3\mathbf{k}\omega_{\mathbf{k}}a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}(2\pi)^3\delta(0)\int đ^3\mathbf{k}\omega_{\mathbf{k}} \end{aligned}$$

In the third equality we use the relations (1) from problem 2(b). In the last equality we use the commutator (2) from problem 2(c), as well as the expressions $a_{\mathbf{k}}(t) = e^{-i\omega_{\mathbf{k}}t}a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger(t) = e^{i\omega_{\mathbf{k}}t}a_{\mathbf{k}}^\dagger$. Note that in the above calculation, we showed that the time-dependence cancels explicitly. We could have also used that H is conserved to remove the time-dependence immediately by evaluating all fields at $t=0$.

This can be written as

$$H = \int đ^3\mathbf{k}\omega_{\mathbf{k}}N_{\mathbf{k}} + E_0$$

for the number operator $N_{\mathbf{k}} = a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$, and zero-point energy $E_0 = \frac{1}{2}(2\pi)^3\delta(0)\int đ^3\mathbf{k}\omega_{\mathbf{k}}$.

(b) Express the conserved charges P^i for spatial translations, in terms of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$.
Again we quote the charges from problem set 1, and use (3) to write it in momentum space.

$$\begin{aligned} P^i &= \int d^3\mathbf{x}\,\pi\partial^i\phi = -i\int đ^3\mathbf{k}\pi(\mathbf{k},t)\phi(-\mathbf{k},t)k^i \\ &= \frac{1}{2}\int đ^3\mathbf{k}k^i\left(a_{\mathbf{k}}(t) - a_{-\mathbf{k}}(t)^\dagger\right)\left(a_{-\mathbf{k}}(t) + a_{\mathbf{k}}(t)^\dagger\right) \\ &= \frac{1}{2}\int đ^3\mathbf{k}k^i\left(a_{\mathbf{k}}a_{-\mathbf{k}}e^{-2i\omega_{\mathbf{k}}t} + a_{\mathbf{k}}a_{\mathbf{k}}^\dagger - a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} - a_{-\mathbf{k}}^\dagger a_{\mathbf{k}}^\dagger e^{2i\omega_{\mathbf{k}}t}\right) \end{aligned}$$

In the third equality we use the relations (1) from problem 2(b). Observe that due to the k^i factor and the commutation relations (2), the first and fourth terms in our final expression are odd under the change of variables $\mathbf{k} \to -\mathbf{k}$, so they must vanish. Therefore,

$$P^i = \frac{1}{2}\int đ^3\mathbf{k}k^i\left(a_{\mathbf{k}}a^\dagger_{\mathbf{k}} + a^\dagger_{\mathbf{k}}a_{\mathbf{k}}\right) = \int đ^3\mathbf{k}k^i a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}\int đ^3\mathbf{k}(2\pi)^3\delta^{(3)}(0)k^i = \int đ^3\mathbf{k}k^i N_{\mathbf{k}}$$

In the second equality we use the commutator (2), and in the third equality we note that the last term vanishes because the integrand is $\mathbf{k}$-odd.

We may combine this with the expression for H in part (a) to write

$$P^\mu = \int đ^3\mathbf{k}\, k^\mu N_{\mathbf{k}} + \delta^{\mu 0}E_0, \qquad k^0 = \omega_{\mathbf{k}} \tag{4}$$

(c) Starting with

$$\phi(0,0) = \int đ^3k\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}\right)$$

show that under the action of translation operators,

$$\phi(\mathbf{x},t) = e^{iHt - iP^ix^i}\phi(0,0)e^{-iHt+iP^ix^i}$$

Hint: this problem becomes trivial using the following formula for a harmonic oscillator,

$$e^{i\alpha N}ae^{-i\alpha N} = e^{-i\alpha}a, \qquad\qquad N = a^\dagger a$$

The formula in the hint follows from the Baker-Campbell-Hausdorff (BCH) formula,

$$e^XYe^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \cdots$$

We check:

$$\begin{aligned}
[i\alpha N, a] &= i\alpha[a^\dagger a, a] = -i\alpha a\\
e^{i\alpha N}ae^{-i\alpha N} &= a + (-i\alpha)a + \frac{1}{2!}(-i\alpha)^2 a + \cdots = e^{-i\alpha}a
\end{aligned}$$

and similarly, $e^{-i\alpha N}a^\dagger e^{-i\alpha N} = e^{i\alpha}a^\dagger$.

Now we generalize. For $\alpha(\mathbf{k}')$ a real-valued function,

$$\begin{aligned}
e^{i\int đ^3\mathbf{k}'\alpha(\mathbf{k}')N_{\mathbf{k}'}}a_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'\alpha(\mathbf{k}')N_{\mathbf{k}'}} &= e^{i\int đ^3\mathbf{k}'\delta(\mathbf{k}-\mathbf{k}')\alpha(\mathbf{k}')N_{\mathbf{k}'}}a_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'\delta(\mathbf{k}-\mathbf{k}')\alpha(\mathbf{k}')N_{\mathbf{k}'}}\\
&= e^{i\alpha(\mathbf{k})N_{\mathbf{k}}}a_{\mathbf{k}}e^{-i\alpha(\mathbf{k})N_{\mathbf{k}}} = e^{-i\alpha(\mathbf{k})}a_{\mathbf{k}}
\end{aligned} \tag{5}$$

In the first equality, we use that the 2 sets of operators $\{\alpha_{\mathbf{k}}, \alpha^\dagger_{\mathbf{k}}, N_{\mathbf{k}}\}$ and $\{\alpha_{\mathbf{k}'}, \alpha^\dagger_{\mathbf{k}'}, N_{\mathbf{k}'}\}$ commute with each other for $\mathbf{k} \neq \mathbf{k}'$. This allows us to move all but the $\mathbf{k}' = \mathbf{k}$ exponentials on the left-hand side past the $a_{\mathbf{k}}$ factor, where it cancels out with the exponentials on the right. Note that for this, it is essential that $\alpha(\mathbf{k}')$ is a real-valued function. The last equality follows from the instance of the BCH formula derived above. In the same way, we have that

$$e^{i\int đ^3\mathbf{k}'\alpha(\mathbf{k}')N_{\mathbf{k}'}}a^\dagger_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'\alpha(\mathbf{k}')N_{\mathbf{k}'}} = e^{i\alpha(\mathbf{k})}a^\dagger_{\mathbf{k}} \tag{6}$$

Using our expressions for H and P^i in part (a) and (b), identities (5)-(6) allow us to compute

$$\begin{aligned}
e^{i(Ht-P^ix^i)}a_{\mathbf{k}}e^{-i(Ht-P^ix^i)} &= e^{i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}+iE_0t}a_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}-iE_0t} \\
&= e^{i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}}a_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}} = e^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}a_{\mathbf{k}} \\
e^{i(Ht-P^ix^i)}a^\dagger_{\mathbf{k}}e^{-i(Ht-P^ix^i)} &= e^{i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}}a^\dagger_{\mathbf{k}}e^{-i\int đ^3\mathbf{k}'(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{x})N_{\mathbf{k}'}} = e^{i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}a^\dagger_{\mathbf{k}}
\end{aligned}$$

Finally, we get

$$\begin{aligned}
e^{i(Ht-P^ix^i)}\phi(0,0)e^{-i(Ht-P^ix^i)} &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}e^{i(Ht-P^ix^i)}(a_{\mathbf{k}}+a^\dagger_{\mathbf{k}})e^{-i(Ht-P^ix^i)} \\
&= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}(a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+a^\dagger_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}) = \phi(\mathbf{x},t)
\end{aligned}$$

Question 4: Noether charges for Lorentz symmetries of a real scalar (20 points + 10 bonus)
In this problem we work out the conserved currents corresponding to the Lorentz symmetries of a real scalar theory,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

(a) Consider an infinitesimal Lorentz transformation

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are infinitesimal numbers. Show that this satisfies

$$\Lambda_\mu{}^\rho\eta_{\rho\lambda}\Lambda_\nu{}^\lambda = \eta_{\mu\nu}$$

to first order in $\omega_{\mu\nu}$, so this does give a Lorentz transformation.

We compute:

$$\begin{aligned}\Lambda_\mu{}^\rho\eta_{\rho\lambda}\Lambda_\nu{}^\lambda &= (\delta_\mu{}^\rho + \omega_\mu{}^\rho)\eta_{\rho\lambda}(\delta_\nu{}^\lambda + \omega_\nu{}^\lambda) = (\eta_{\mu\lambda} + \omega_{\mu\lambda})(\delta^\lambda{}_\nu - \omega^\lambda{}_\nu) \\ &= \eta_{\mu\nu} + \omega_{\mu\nu} - \omega_{\mu\nu} + \omega_{\mu\lambda}\omega^\lambda{}_\mu = \eta_{\mu\nu} + \mathcal{O}(\omega^2)\end{aligned}$$

(b) Write down how ϕ transforms under an infinitesimal Lorentz transformation, and show that the conserved Noether current for this transformation can be written as

$$J^{\mu\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}$$

where $T^{\mu\nu}$ is the conserved energy-momentum tensor derived in problem set 1.

A Lorentz scalar field transforms in a way obeying $\phi'(x') = \phi(x)$. Therefore, under an infinitesimal Lorentz transformation, the scalar field ϕ transforms as

$$\delta\phi = \phi'(x) - \phi(x) = \phi((\Lambda^{-1})^\mu{}_\nu x^\nu) - \phi(x^\nu) = \phi((\delta^\mu{}_\nu - \omega^\mu{}_\nu)x^\nu) - \phi(x^\nu) = -\omega^\mu{}_\nu x^\nu\partial_\mu\phi$$

where in the last equality we Taylor expand $\phi(x^\mu - \omega^\mu{}_\nu x^\nu) = -\omega^\mu{}_\nu x^\nu\partial_\mu\phi$.

Using this, the Lagrangian density transforms as

$$\delta\mathcal{L} = \mathcal{L}[\phi'] - \mathcal{L}[\phi] = \mathcal{L}[\phi - \omega^\lambda{}_\nu x^\nu\partial_\lambda\phi] - \mathcal{L}[\phi] = -\omega^\lambda{}_\nu x^\nu\partial_\lambda\phi\frac{\partial\mathcal{L}}{\partial\phi} = -\omega^\lambda{}_\nu x^\nu\partial_\lambda\mathcal{L} = -\partial_\lambda(\omega^\lambda{}_\nu x^\nu\mathcal{L})$$

We expand only to first order in ω. In the final equality, we use that $\omega_{\mu\nu}$ is antisymmetric, i.e.

$$\partial_\lambda(\omega^\lambda{}_\nu x^\nu f(x)) = \omega^\lambda{}_\nu\delta^\nu_\lambda f(x) + \omega^\lambda{}_\nu x^\nu\partial_\lambda f(x) = \omega^\lambda{}_\lambda f(x) + \omega^\lambda{}_\nu x^\nu\partial_\lambda f(x) = \omega^\lambda{}_\nu x^\nu\partial_\lambda f(x)$$

Hence, $\delta\mathcal{L} = \partial_\mu\mathcal{F}^\mu$, for $\mathcal{F}^\mu = -\omega^\mu{}_\nu x^\nu\mathcal{L}$

The Noether current for the transformation parameterized by $\omega_{\lambda\nu}$ is given by

$$\begin{aligned}j^\mu &= -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - \mathcal{F}^\mu = \omega^\lambda{}_\nu x^\nu\partial^\mu\phi\partial_\lambda\phi + \omega^\mu{}_\nu x^\nu\mathcal{L} \\ &= \omega_{\lambda\nu}x^\nu(\partial^\mu\phi\partial^\lambda\phi + \eta^{\mu\lambda}\mathcal{L}) = \omega_{\lambda\nu}x^\nu T^{\mu\lambda}\end{aligned}$$

for the energy-momentum tensor from problem set 1 (now with a real scalar):

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(\partial_\rho\phi\partial^\rho\phi + m^2\phi^2)$$

Note that $\omega_{\lambda\nu}$ is an arbitrary antisymmetric tensor which parameterizes our infinitesimal transformation. In total we have an antisymmetric tensor worth of conserved currents, which we can package in $J^{\mu\lambda\nu}$:

$$J^{\mu\lambda\nu} = x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}, \qquad j^\mu = -\frac{1}{2}\omega_{\lambda\nu}J^{\mu\lambda\nu}$$

In writing $J^{\mu\lambda\nu}$ we have made the antisymmetry in λ and ν manifest by explicitly antisymmetrizing over these indices.

(c) Using conservation of the energy-momentum tensor, verify that the current in (b) is conserved, i.e.

$$\partial_\mu J^{\mu\lambda\nu} = 0$$

We compute:

$$\begin{aligned}\partial_\mu J^{\mu\lambda\nu} &= \partial_\mu(x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}) = \delta_\mu{}^\lambda T^{\mu\nu} + x^\lambda \partial_\mu T^{\mu\nu} - \delta_\mu{}^\nu T^{\mu\lambda} - x^\nu \partial_\mu T^{\mu\lambda} \\ &= T^{\lambda\nu} - T^{\nu\lambda} = 0\end{aligned}$$

In the 3rd equality we used the conservation law $\partial_\mu T^{\mu\nu} = 0$, and in the 4th equality we used from problem set 1 that $T^{\mu\nu} = T^{\nu\mu}$ is symmetric.

(d) (**Bonus problem**) Consider the conserved charges associated with $J^{\mu\lambda\nu}$,

$$M^{\lambda\nu} = \int d^3x J^{0\lambda\nu}$$

Express the conserved charges $M^{\mu\nu}$ for the Lorentz symmetries of this theory in terms of $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$.
From part (b), we have

$$M^{\mu\nu} = \int d^3x J^{0\mu\nu} = \int d^3x(x^\mu T^{0\nu} - x^\nu T^{0\mu})$$

This is antisymmetric in μ and ν, so we need to compute M^{0i} and M^{ij}. To do this, we need to expand $T^{0\mu}$ in terms of creation and annihilation operators:

$$\begin{aligned}T^{0\mu} &= -\pi\partial^\mu\phi - \frac{1}{2}\eta^{0\mu}\left(-\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right) \\ &= \left(\frac{1}{2}(\pi^2 + (\nabla\phi)^2 + m^2\phi^2), -\pi\partial^i\phi\right) = (\mathcal{H}(x), \mathcal{P}^i(x))\end{aligned}$$

Since $M^{\mu\nu}$ are conserved currents, we can compute them at $t = 0$.
First, we need the identity

$$\begin{aligned}\int d^3\mathbf{x} x^i f(\mathbf{x})g(\mathbf{x}) &= \int d^3\mathbf{x} d^3\mathbf{k} d^3\mathbf{k}' e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} x^i f(\mathbf{k})g(\mathbf{k}') \\ &= \frac{1}{(2\pi)^3}\int d^3\mathbf{k} d^3\mathbf{k}'(-i\partial_{k_i}\delta^{(3)}(\mathbf{k}+\mathbf{k}'))f(\mathbf{k})g(\mathbf{k}') = i\int đ^3\mathbf{k}\partial_{k_i} f(\mathbf{k})g(-\mathbf{k}) \qquad (7)\end{aligned}$$

where we have used integration by parts in the last equality. More generally if there are derivatives acting on f or g, each derivative acting on f drags down a factor of $i\mathbf{k}$, while each derivative actin on g drags down a factor of $i\mathbf{k}'$, which becomes $-i\mathbf{k}$ after performing the $d^3\mathbf{x}$ integral.

Now we are ready to compute

$$
\begin{aligned}
M^{0i} &= \int d^3\mathbf{x}(t\mathcal{P}^i(x) - x^i\mathcal{H}(x))|_{t=0} = -\frac{1}{2}\int d^3\mathbf{x}x^i(\pi^2 + (\nabla\phi)^2 + m^2\phi^2) \\
&= -\frac{i}{2}\int \bar{d}^3\mathbf{k}\left(\partial_{k_i}\pi(\mathbf{k})\pi(-\mathbf{k}) + \omega_{\mathbf{k}}^2\partial_{k_i}\phi(\mathbf{k})\phi(-\mathbf{k})\right) \\
&= -\frac{i}{4}\int \bar{d}^3\mathbf{k}\Big(-\partial_{k_i}(\sqrt{\omega_{\mathbf{k}}}(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}))\cdot(\sqrt{\omega_{\mathbf{k}}}(a_{-\mathbf{k}} - a^\dagger_{\mathbf{k}})) \\
&\qquad\qquad + \omega_{\mathbf{k}}^2\partial_{k_i}(\frac{1}{\sqrt{\omega_{\mathbf{k}}}}(a_{\mathbf{k}} + a^\dagger_{-\mathbf{k}}))\cdot(\frac{1}{\sqrt{\omega_{\mathbf{k}}}}(a_{-\mathbf{k}} + a^\dagger_{\mathbf{k}}))\Big) \\
&= +\frac{i}{4}\int \bar{d}^3\mathbf{k}\left((k^i + \omega_{\mathbf{k}}\partial_{k_i})(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}})\cdot(a_{-\mathbf{k}} - a^\dagger_{\mathbf{k}}) + (k^i - \omega_{\mathbf{k}}\partial_{k_i})(a_{\mathbf{k}} + a^\dagger_{-\mathbf{k}})\cdot(a_{-\mathbf{k}} + a^\dagger_{\mathbf{k}})\right) \\
&= -\frac{i}{2}\int \bar{d}^3\mathbf{k}\omega_{\mathbf{k}}\left((\partial_{k_i}a_{\mathbf{k}})a^\dagger_{\mathbf{k}} + (\partial_{k_i}a^\dagger_{-\mathbf{k}})a_{-\mathbf{k}}\right) = -\frac{i}{2}\int \bar{d}^3\mathbf{k}\omega_{\mathbf{k}}\left((\partial_{k_i}a_{\mathbf{k}})a^\dagger_{\mathbf{k}} - (\partial_{k_i}a^\dagger_{\mathbf{k}})a_{\mathbf{k}}\right) \\
&= -\frac{i}{2}\int \bar{d}^3\mathbf{k}\omega_{\mathbf{k}}\left(a^\dagger_{\mathbf{k}}(\partial_{k_i}a_{\mathbf{k}}) + (2\pi)^3\partial_{k_i}\delta(\mathbf{k}-\mathbf{k}')|_{\mathbf{k}'=\mathbf{k}} - (\partial_{k_i}a^\dagger_{\mathbf{k}})a_{\mathbf{k}}\right) \\
&= -\frac{i}{2}\int \bar{d}^3\mathbf{k}\omega_{\mathbf{k}}\left(a^\dagger_{\mathbf{k}}(\partial_{k_i}a_{\mathbf{k}}) - (\partial_{k_i}a^\dagger_{\mathbf{k}})a_{\mathbf{k}}\right)
\end{aligned}
$$

Line 2 follows from identity (7). Line 5 is obtained by noting that all but 2 terms in the integrand of line 4 either cancel out or are odd. In line 6 we use the identity obtained by taking the $\mathbf{k}$-derivative of $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$, and evaluating at $\mathbf{k}' = \mathbf{k}$. Finally, to reach line 7 we use that the $\delta^{(3)}$-term in line 6 is odd.

In a similar way, we can also compute for $i \neq j$ (since $M^{ii} = 0$ by asymmetry)

$$
\begin{aligned}
M^{ij} &= \int d^3\mathbf{x}(x^i\mathcal{P}^j(x) - x^j\mathcal{P}^i(x)) = -\int d^3\mathbf{x}(x^i\pi\partial^j\phi - x^j\pi\partial^i\phi) \\
&= \frac{1}{2}\int \bar{d}^3\mathbf{k}\left(k^j\pi(\mathbf{k})\partial_{k_i}\phi(-\mathbf{k}) - (i\leftrightarrow j)\right) \\
&= -\frac{i}{4}\int \bar{d}^3\mathbf{k}\left(k^j\sqrt{\omega_{\mathbf{k}}}(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}})\cdot\partial_{k_i}(\frac{1}{\sqrt{\omega_{\mathbf{k}}}}(a_{-\mathbf{k}} + a^\dagger_{\mathbf{k}})) - (i\leftrightarrow j)\right) \\
&= -\frac{i}{4}\int \bar{d}^3\mathbf{k}\left(k^j(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}})\cdot\left(-\frac{k^i}{\omega_{\mathbf{k}}} + \partial_{k_i}\right)(a_{-\mathbf{k}} + a^\dagger_{\mathbf{k}}) - (i\leftrightarrow j)\right) \\
&= -\frac{i}{4}\int \bar{d}^3\mathbf{k}k^j\left(a_{\mathbf{k}}\partial_{k_i}a_{-\mathbf{k}} + a_{\mathbf{k}}\partial_{k_i}a^\dagger_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}\partial_{k_i}a_{-\mathbf{k}} - a^\dagger_{-\mathbf{k}}\partial_{k_i}a^\dagger_{\mathbf{k}}\right) - (i\leftrightarrow j) \\
&= -\frac{i}{4}\int \bar{d}^3\mathbf{k}k^j\left(a_{\mathbf{k}}\partial_{k_i}a^\dagger_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}\partial_{k_i}a_{-\mathbf{k}}\right) - (i\leftrightarrow j) \\
&= -\frac{i}{4}\int \bar{d}^3\mathbf{k}k^j\left(-(\partial_{k_i}a_{\mathbf{k}})a^\dagger_{\mathbf{k}} - a^\dagger_{\mathbf{k}}(\partial_{k_i}a_{\mathbf{k}})\right) - (i\leftrightarrow j) \\
&= -\frac{i}{4}\int \bar{d}^3\mathbf{k}k^j\left((2\pi)^3\partial_{k_i}\delta(\mathbf{k}-\mathbf{k}')|_{\mathbf{k}'=\mathbf{k}} - 2a^\dagger_{\mathbf{k}}(\partial_{k_i}a_{\mathbf{k}})\right) - (i\leftrightarrow j) \\
&= -\frac{i}{2}\int \bar{d}^3\mathbf{k}\left(k^i a^\dagger_{\mathbf{k}}(\partial_{k_j}a_{\mathbf{k}}) - k^j(\partial_{k_i}a^\dagger_{\mathbf{k}})a_{\mathbf{k}}\right)
\end{aligned}
$$

where we make ample use of integration by parts, drop total-derivative terms, and use $i \neq j$ to completely ignore having to deal with $\propto \partial_{k_i}k^j$ terms.

Physically, the conserved quantities are the center of mass velocities M^{0i} and angular momenta M^{ij}.

Altogether, we may write

$$M^{\mu\nu} = -\frac{i}{2}\int đ^3\mathbf{k}k^\mu(a^\dagger_{\mathbf{k}}\partial_{k_\nu}a_{\mathbf{k}} - \partial_{k_\nu}a^\dagger_{\mathbf{k}}a_{\mathbf{k}}) - (\mu \to \nu), \qquad\qquad k^0 = \omega_{\mathbf{k}}$$

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