8.323 Problem Set 1 Solutions

February 15, 2023

Question 1: Quantum harmonic oscillator in the Heisenberg picture (25 points)
Consider the Hamiltonian for a unit mass harmonic oscillator with frequency ω,

$$H = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2)$$

In the Heisenberg picture $\hat{p}(t)$ and $\hat{x}(t)$ are dynamical variables which evolve with time. They obey the equal-time commutation relation

$$[\hat{x}(t), \hat{p}(t)] = i$$

Here and below we set $\hbar = 1$.

(a) Obtain the Heisenberg evolution equations for $\hat{x}(t)$ and $\hat{p}(t)$.
We use Heisenberg's equations of motion for $\hat{x}(t)$ and $\hat{p}(t)$:

$$\frac{d\hat{x}(t)}{dt} = i[H, \hat{x}(t)] \qquad \frac{d\hat{p}(t)}{dt} = i[H, \hat{p}(t)]$$

The right hand sides can be computed using $H = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2)$, the commutator $[\hat{x}, \hat{p}] = i$, and the Heisenberg time evolution $\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt}$. For instance:

$$[H, \hat{x}(t)] = [H, e^{iHt}\hat{x}e^{-iHt}] = e^{iHt}[H, \hat{x}]e^{-iHt} = -ie^{iHt}\hat{p}e^{-iHt} = -i\hat{p}(t)$$

Hence, we have:

$$\frac{d\hat{x}(t)}{dt} = \hat{p}(t) \qquad \frac{d\hat{p}(t)}{dt} = -\omega^2 \hat{x}(t)$$

(b) Suppose the initial conditions at $t = 0$ are given by

$$\hat{x}(0) = \hat{x} \qquad \hat{p}(0) = \hat{p}$$

Find $\hat{x}(t)$ and $\hat{p}(t)$.
We can decouple the system by converting to second order equations:

$$\ddot{\hat{x}} = -\omega^2 \hat{x}(t) \qquad \ddot{\hat{p}}(t) = -\omega^2 \hat{p}(t)$$

Solving with the initial conditions $\hat{x}(0) = \hat{x}$ and $\hat{p}(0) = \hat{p}$, we find

$$\hat{x}(t) = \hat{x}\cos\omega t + \frac{1}{\omega}\hat{p}\sin\omega t \qquad \hat{p}(t) = \hat{p}\cos\omega t - \omega\hat{x}\sin\omega t$$

(c) It is convenient to introduce operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ defined by:

$$\hat{x}(t) = \sqrt{\frac{1}{2\omega}}(\hat{a}(t) + \hat{a}^\dagger(t)), \qquad \hat{p}(t) = -i\sqrt{\frac{\omega}{2}}(\hat{a}(t) - \hat{a}^\dagger(t))$$

Show that $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ satisfy the equal-time commutation relation

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1$$

We solve for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ in terms of $\hat{x}(t)$ and $\hat{p}(t)$:

$$\hat{a}(t) = \sqrt{\frac{\omega}{2}}\hat{x}(t) + i\sqrt{\frac{1}{2\omega}}\hat{p}(t), \qquad \hat{a}^\dagger(t) = \sqrt{\frac{\omega}{2}}\hat{x}(t) - i\sqrt{\frac{1}{2\omega}}\hat{p}(t)$$

Using the commutation relations between position and momentum operators, we have

$$[\hat{a}(t), \hat{a}^\dagger(t)] = -\frac{i}{2}[\hat{x}(t), \hat{p}(t)] + \frac{i}{2}[\hat{p}(t), \hat{x}(t)] = 1$$

(d) Express the Hamiltonian in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

$$\begin{aligned} H = H(t) = e^{iHt}He^{-iHt} &= \frac{1}{2}(\hat{p}(t)^2 + \omega^2\hat{x}(t)^2) \\ &= \frac{\omega}{4}(-(\hat{a}(t) - \hat{a}^\dagger(t))^2 + (\hat{a}(t) - \hat{a}^\dagger(t))^2) = \frac{\omega}{2}(\hat{a}(t)\hat{a}^\dagger(t) + \hat{a}^\dagger(t)\hat{a}(t)) \\ &= \omega\left(\hat{a}^\dagger(t)\hat{a}(t) + \frac{1}{2}\right) = \omega\left(N(t) + \frac{1}{2}\right) \end{aligned}$$

where in the last equality we define the number operator, $N(t) = \hat{a}^\dagger(t)\hat{a}(t)$.

(e) Obtain the Heisenberg equations for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.
Using the results in parts (c) and (d), we have

$$\begin{aligned} \frac{d\hat{a}(t)}{dt} &= i[H, \hat{a}(t)] = i\omega[\hat{a}^\dagger(t)\hat{a}(t), \hat{a}(t)] = -i\omega\hat{a}(t) \\ \frac{d\hat{a}^\dagger(t)}{dt} &= i[H, \hat{a}^\dagger(t)] = i\omega[\hat{a}^\dagger(t)\hat{a}(t), \hat{a}^\dagger(t)] = i\omega\hat{a}^\dagger(t) \end{aligned}$$

(f) Suppose the initial conditions at $t = 0$ are given by

$$\hat{a}(0) = \hat{a}, \qquad \hat{a}^\dagger(0) = \hat{a}^\dagger$$

Find $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.
The equations in (e) are decoupled, and first-order linear. We immediately have

$$\hat{a}(t) = \hat{a}e^{-i\omega t}, \qquad \hat{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega t}$$

(g) Express $\hat{x}(t)$, $\hat{p}(t)$, and the Hamiltonian H in terms of $\hat{a}$ and $\hat{a}^\dagger$.
We substitute the expressions obtained in (f) into parts (c) and (d).

$$\begin{aligned} \hat{x}(t) &= \sqrt{\frac{1}{2\omega}}(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}) \\ \hat{p}(t) &= -i\sqrt{\frac{\omega}{2}}(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}) \\ H(t) = H &= \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \end{aligned}$$

Question 2: Lorentz transformations (15 points)

(a) Probe that the 4-dimensional δ-function

$$\delta^{(4)}(p) = \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3)$$

is Lorentz invariant, i.e.

$$\delta^{(4)}(p) = \delta^{(4)}(\tilde{p})$$

where $\tilde{p}^\mu$ is a Lorentz transformation of p^μ.

We express the δ-function in integral form, and use that $p \cdot x$ is a Lorentz scalar, i.e. $\Lambda p \cdot \Lambda x = p \cdot x$.

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4}\int d^4x e^{ip\cdot x} = \frac{1}{(2\pi)^4}\int d^4x e^{i\Lambda p\cdot\Lambda x}$$

Now we make the change of variables $\tilde{x} = \Lambda x$. Note that $d^4\tilde{x} = d^4x$. To see this we use $\Lambda^T\eta\Lambda = \eta$, which implies $1 = \det(\Lambda^T)\det(\Lambda) = (\det\Lambda)^2$. Hence, the Jacobian $J = |\det\Lambda| = 1$. One thus has

$$\delta^{(4)}(p) = \frac{1}{(2\pi)^4}\int d^4\tilde{x} e^{i\Lambda p\cdot\tilde{x}} = \frac{1}{(2\pi)^4}\int d^4x e^{i\Lambda p\cdot x} = \delta^{(4)}(\Lambda p)$$

(b) Show that

$$\omega_1\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$$

is Lorentz invariant, i.e.

$$\omega_1\delta^{(3)}(\vec{k}_1 - \vec{k}_2) = \omega_1'\delta^{(3)}(\vec{k}_1' - \vec{k}_2')$$

Here $\vec{k}_1$ and $\vec{k}_2$ are respectively the spatial part of four-vectors $k_1^\mu = (\omega_1, \vec{k}_1)$ and $k_2^\mu = (\omega_2, \vec{k}_2)$ which satisfy the on-shell condition

$$k_1^2 + k_2^2 = -m^2$$

and $k_1'^\mu = (\omega_1', \vec{k}_1')$, $k_2'^\mu = (\omega_2', \vec{k}_2')$ are related to k_1^μ, k_2^μ by the same Lorentz transformation.

We consider the expression $\delta(k^2 + m^2)$ which imposes the mass-shell constraint. We can simplify this using the δ-function identity $\delta(f(x)) = \sum_{x_i \text{ s.t. } f(x_i)=0} \frac{1}{|f'(x_i)|}\delta(x - x_i)$.

$$\delta(k^2 + m^2) = \delta(-k_0^2 + \vec{k}^2 + m^2) = \frac{1}{2|\omega_{\vec{k}}|}\left(\delta(k_0 - |\omega_{\vec{k}}|) + \delta(k_0 + |\omega_{\vec{k}}|)\right) \tag{1}$$

We will assume $\omega_{\vec{k}_1}, \omega_{\vec{k}_2} > 0$, as is the case for physical 4-momenta.

We can pick out the $k_1^0 = \omega_{\vec{k}_1}$ enforcing δ-function in (1) by multiplying both sides by $\theta(\omega_{\vec{k}_1})$.

$$\theta(\omega_{\vec{k}_1})\delta(k_1^2 + m^2) = \frac{1}{2\omega_{\vec{k}_1}}\delta(k_1^0 - \omega_{\vec{k}_1})\theta(\omega_{\vec{k}_1}) = \frac{1}{2\omega_{\vec{k}_1}}\delta(k_1^0 - \omega_{\vec{k}_1}) \tag{2}$$

Now we multiply both sides by $\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$:

$$\begin{aligned}
\theta(\omega_{\vec{k}_1})\delta(k_1^2 + m^2)\cdot 2\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2) &= \delta(k_1^0 - \omega_{\vec{k}_1})\delta^{(3)}(\vec{k}_1 - \vec{k}_2) \\
&= \delta(k_1^0 - \omega_{\vec{k}_2})\delta^{(3)}(\vec{k}_1 - \vec{k}_2) \\
&= \delta(k_1^0 - k_2^0)\delta^{(3)}(\vec{k}_1 - \vec{k}_2) \\
\theta(\omega_{\vec{k}_1})\delta(k_1^2 + m^2)\cdot 2\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2) &= \delta^{(4)}(k_1^\mu - k_2^\mu) \qquad (3)
\end{aligned}$$

In the second equality, we use that the $\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$ allows us to replace $\omega_{\vec{k}_1}$ with $\omega_{\vec{k}_2}$. For this step, it is crucial that $\mathrm{sign}(\omega_{\vec{k}_1}) = \mathrm{sign}(\omega_{\vec{k}_2})$, which is true since both are positive.

Finally, let us study (3). The right-hand side is Lorentz invariant by part (a). On the left-hand side, $\delta(k_1^2 + m^2)$ is Lorentz invariant since k_1^2 is a Lorentz scalar, and $\theta(\omega_{\vec{k}_1})$ is Lorentz invariant because the energy of a particle does not change under a (proper, orthochronous) Lorentz transformation. It then follows that $\omega_{\vec{k}_1}\delta^{(3)}(\vec{k}_1 - \vec{k}_2)$ is Lorentz invariant.

(c) For any function $f(k) = f(k^0, k^1, k^2, k^3)$, prove that

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k), \qquad \omega_{\vec{k}} = \sqrt{\vec{k^2} + m^2}$$

is Lorentz invariant, in the sense that

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\tilde{k})$$

where $\tilde{k}^\mu = \Lambda^\mu{}_\nu k^\nu$ is a Lorentz transformation of k^μ.

Since the momentum is on the mass-shell, we write $f(k) = f(\omega_{\vec{k}}, \vec{k})$.
By introducing another δ-function, we may write this expression as a integral over 4-dimensions:

$$\begin{aligned}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\omega_{\vec{k}}, \vec{k}) &= \frac{1}{(2\pi)^3}\int d^4k \frac{1}{2\omega_{\vec{k}}}\delta(k^0 - \omega_{\vec{k}})f(k^0, \vec{k}) \\ &= \frac{1}{(2\pi)^3}\int d^4k\, \theta(\omega_{\vec{k}})\delta(k^2 + m^2)f(k^\mu) \end{aligned} \tag{4}$$

where in the last equality we have used (2).
Now we make a change of variables, $k = \Lambda k'$, for Λ an arbitrary (proper, orthochronous) Lorentz transformation. In parts (a)-(b), we showed that $d^4k = d^4k'$, $\theta(\omega_{\vec{k}}) = \theta(\omega_{\vec{k}'})$, and $\delta(k^2 + m^2) = \delta(k'^2 + m^2)$. Hence,

$$\begin{aligned}\frac{1}{(2\pi)^3}\int d^4k\, \theta(\omega_{\vec{k}})\delta(k^2 + m^2)f(k^\mu) &= \frac{1}{(2\pi)^3}\int d^4k'\, \theta(\omega_{\vec{k}'})\delta(k'^2 + m^2)f((\Lambda k')^\mu) \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\Lambda k)\end{aligned}$$

where the last equality is obtained using the reverse sequence of operations that led to (4).
Putting everything together, we have the desired result,

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}f(\Lambda k)$$

Question 3: A complex scalar field (20 points)
Consider the field theory of a complex valued scalar field $\phi(x)$ with action

$$S = \int d^4x\big(- \partial_\mu\phi^*\partial^\mu\phi - V(|\phi|^2)\big), \qquad |\phi|^2 = \phi^*\phi$$

One could either consider the real and imaginary parts of ϕ, or ϕ and ϕ^* as independent dynamical variables. The latter is more convenient in most situations.
(a) Check that the action is Lorentz invariant, and find the equations of motion.
A Lorentz transformation acts as $\phi \to \phi'$, such that $\phi'(x) = \phi(\Lambda^{-1}x)$. The action transforms as:

$$\begin{aligned} S \to S' &= \int d^4x\big(- \partial_\mu\phi'^*(x)\partial^\mu\phi'(x) - V(|\phi'(x)|^2)\big) \\ &= \int d^4x\big(- \partial_\mu\phi^*(\Lambda^{-1}x)\partial^\mu\phi(\Lambda^{-1}x) - V(|\phi(\Lambda^{-1}x)|^2)\big) \end{aligned}$$

Now we make the change of variable $x' = \Lambda^{-1}x$. We showed in 2(a) that $d^4x = d^4x'$. Furthermore, by the chain rule we have $\partial_\mu = (\Lambda^{-1})_\mu{}^\nu\partial'_\nu$. Here ∂_μ and ∂'_μ denote differentiation with respect to x and x'.

$$\begin{aligned} S' &= \int d^4x'\big(- (\Lambda^{-1})_\mu{}^\nu\partial'_\nu\phi^*(x')\ (\Lambda^{-1})^\mu{}_\rho\partial'^\rho\phi(x') - V(|\phi(x')|^2\big) \\ &= \int d^4x'\big(- \delta^\nu_\rho\partial'_\nu\phi^*(x')\partial'^\rho\phi(x') - V(|\phi(x')|^2\big) \\ &= \int d^4x'\big(- \partial'_\nu\phi^*(x')\partial'^\nu\phi(x') - V(|\phi(x')|^2\big) = S \end{aligned}$$

where in the second line we use

$$(\Lambda^{-1})_\mu{}^\nu(\Lambda^{-1})^\mu{}_\rho = ((\Lambda^{-1})^T)^\nu{}_\mu(\Lambda^{-1})^\mu{}_\rho = (\Lambda)^\nu{}_\mu(\Lambda^{-1})^\mu{}_\rho = \delta^\nu_\rho$$

To find the equations of motion, we use the Euler-Lagrange equations, treating ϕ and ϕ^* as independent:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi}, \qquad \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = \frac{\partial\mathcal{L}}{\partial\phi^*}$$

The left-hand side can be confusing to evaluate due to the contracted indices, so we do one calculation very explicitly:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\left[\frac{\partial}{\partial(\partial_\mu\phi)}(-\partial_\nu\phi^*\partial^\nu\phi)\right] = \partial_\mu[-\partial_\nu\phi^*\delta^\nu_\mu] = -\partial^2\phi^*$$

Hence, the equations of motion are

$$\partial^2\phi^* - V'(|\phi|^2)\phi^* = 0, \qquad \partial^2\phi - V'(|\phi|^2)\phi = 0$$

Note that these are conjugate equations, as expected.

(b) Find the canonical conjugate momenta for ϕ and ϕ^*, and the Hamiltonian H.
We write the Lagrangian density as

$$\mathcal{L} = \partial_t\phi^*\partial_t\phi - \vec{\nabla}\phi^*\cdot\vec{\nabla}\phi - V(|\phi|^2)$$

The conjugate momenta are thus

$$\pi := \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi^*, \qquad \pi^* := \frac{\partial\mathcal{L}}{\partial(\partial_t\phi^*)} = \partial_t\phi$$

The Hamiltonian is given by

$$H = \int d^3x(\pi\partial_t\phi + \pi^*\partial_t\phi - \mathcal{L}) = \int d^3x(\pi^*\pi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))$$

(c) The action is invariant under the transformation

$$\phi \to e^{i\alpha}\phi, \qquad \phi^* \to e^{-i\alpha}\phi^*$$

for arbitrary constant α. When α is small, i.e. for an infinitesimal transformation, this becomes

$$\delta\phi = i\alpha\phi, \qquad \delta\phi^* = -i\alpha\phi^*$$

Use Noether's theorem to find the corresponding conserved current j^μ and conserved charge Q.
By Noether's theorem, the conserved current is given by

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_a)}\delta\Phi_a - \mathcal{F}^\mu, \qquad \delta\mathcal{L} = \partial_\mu\mathcal{F}^\mu$$

In this case, $\delta\phi = i\alpha\phi$, $\delta\phi^* = -i\alpha\phi^*$, and $\delta\mathcal{L} = 0$. We find

$$j^\mu = -\partial^\mu\phi^*(i\alpha\phi) - \partial^\mu\phi(-i\alpha\phi^*) = i\alpha(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$$

One may remove the proportionality constant if desired, to get

$$j^\mu = \phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*$$

The corresponding charge is then

$$Q = \int d^3x j^0 = \int d^3x(\phi^*\partial_t\phi - \phi\partial_t\phi^*)$$

(d) Use the equations of motion from part (a) to verify directly that j^μ is conserved.
We compute:

$$\begin{aligned}\partial_\mu j^\mu = \partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) &= \phi^*\partial^2\phi - \phi\partial^2\phi^* \\ &= V'(|\phi|^2)\phi^*\phi - V'(|\phi|^2)\phi^*\phi = 0\end{aligned}$$

where in the second line we use the equations of motion from part (a).

Question 4: The energy-momentum tensor (20 points)

In this problem we work out the energy-momentum tensor of the complex scalar theory in Question 3.

(a) Under a spacetime translation

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

a scalar field transforms as

$$\phi'(x') = \phi(x)$$

Show that the action is invariant under the transformation $\phi(x) \to \phi'(x)$.

Under the transformation, the scalar field satisfies $\phi'(x) = \phi(x-a)$. The action transforms as:

$$\begin{aligned} S \to S' &= \int d^4x \big(-\partial_\mu \phi'^*(x) \partial^\mu \phi'(x) - V(|\phi'(x)|^2) \big) \\ &= \int d^4x \big(-\partial_\mu \phi^*(x-a) \partial^\mu \phi(x-a) - V(|\phi(x-a)|^2) \big) \\ &= \int d^4x \big(-\partial_\mu \phi^*(x) \partial^\mu \phi(x) - V(|\phi(x)|^2) \big) = S \end{aligned}$$

where in the last line we change variables from $x^\mu \to x^\mu + a^\mu$, which does not change the integration measure.

(b) Write down the transformation of the scalar fields ϕ and ϕ^* for an infinitesimal translation, and use Noether's theorem to find the corresponding conserved currents $T^{\mu\nu}$.

An infinitesimal translation acts on the fields as:

$$\begin{aligned} \delta\phi &= \phi'(x) - \phi(x) = \phi(x-a) - \phi(x) = -a^\mu \partial_\mu \phi(x) \\ \delta\phi^* &= \phi'^*(x) - \phi^*(x) = \phi^*(x-a) - \phi^*(x) = -a^\mu \partial_\mu \phi^*(x) \end{aligned}$$

We also need the change in the Lagrangian density under translations:

$$\begin{aligned} \delta\mathcal{L} = \mathcal{L}' - \mathcal{L} &= -\partial_\nu(\phi^*(x) - a^\mu \partial_\mu \phi^*(x))\ \partial^\nu(\phi(x) - a^\mu \partial_\mu \phi(x)) \\ &\quad - V\big((\phi^*(x) - a^\mu \partial_\mu \phi^*(x))(\phi(x) - a^\mu \partial_\mu \phi(x))\big) - \mathcal{L} \\ &= a^\mu(\partial_\nu \partial_\mu \phi^*(x) \partial^\nu \phi(x) + \partial_\nu \phi^*(x) \partial^\nu \partial_\mu \phi(x)) + a^\mu V'(\phi^*\phi)\big(\partial_\mu \phi^* \phi + \phi^* \partial_\mu \phi\big) + \mathcal{O}(a^\mu a^\nu) \\ &= -a^\mu \partial_\mu \mathcal{L} = a_\mu \partial_\nu(-\eta^{\mu\nu}\mathcal{L}) := (a_\mu \partial_\nu)\mathcal{F}^{\mu\nu} \end{aligned}$$

The translations are parameterized by a 4-vector a^μ, and we have a Noether current (itself a 4-vector) for each. Hence, we can encode the conserved currents from translations into a rank-2 tensor, $T^{\mu\nu}$. In the following, we let the first index pick out the direction of the translation a^μ.

The Noether current is

$$\begin{aligned} T^{\mu\nu} := (j^\mu)^\nu &= \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}(\delta\phi)^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi^*)}(\delta\phi^*)^\mu - \mathcal{F}^{\mu\nu} \\ &= -\partial^\nu\phi^*(-\partial^\mu\phi) - \partial^\nu\phi(-\partial^\mu\phi^*) + \eta^{\mu\nu}\mathcal{L} \\ &= \partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi - \eta^{\mu\nu}\big(\partial_\rho\phi^*\partial^\rho\phi + V(|\phi|^2)\big) \end{aligned}$$

(c) The conserved charge for a time translation

$$H = \int d^3x T^{00}$$

should be identified with the total energy of the system, while that for a spatial translation

$$P^i = \int d^3x T^{0i}$$

is identified with the total momentum. Thus $T^{\mu\nu}$ is referred to as the energy-momentum tensor. Write down the explict expressions for H and P^i. Compare H obtained here with the Hamiltonian in problem 3(b).

We compute:

$$\begin{aligned}
H &= \int d^3x T^{00} = \int d^3x \left(2\partial^t\phi^*\partial^t\phi + (-\partial^t\phi^*\partial^t\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))\right) \\
&= \int d^3x \left(\partial_t\phi^*\partial_t\phi + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi + V(|\phi|^2))\right) \\
P^i &= \int d^3x T^{0i} = \int d^3x \left(\partial^t\phi^*\partial^i\phi + \partial^i\phi^*\partial^t\phi\right) = -\int d^3x \left(\partial_t\phi^*\partial_i\phi + \partial_i\phi^*\partial_t\phi\right)
\end{aligned}$$

The expression for the Hamiltonian is equal to the Hamiltonian obtained in problem 3(b).

(d) Use the equations of motion of problem 3(a) to verify directly that $T^{\mu\nu}$ is conserved.

Recall that the first index of $T^{\mu\nu}$ picks out the direction of the translation a^μ, so formally Noether conservation should tell us $\partial_\nu T^{\mu\nu} = 0$. However, from part (c) it can be seen that $T^{\mu\nu}$ is symmetric, so we can contract the derivative with respect to either index.

We compute:

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu(\partial^\mu\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^\mu\phi) - \partial^\nu(\partial_\rho\phi^*\partial^\rho\phi) - \partial^\nu V(|\phi|^2) \\
&= \partial^2\phi^*\partial^\nu\phi + \partial^\nu\phi^*\partial^2\phi - \partial^\nu V(|\phi|^2) \\
&= \phi^*\partial^\nu\phi\ V'(|\phi|^2) + \phi\partial^\nu\phi^*\ V'(|\phi|^2) - \phi^*\partial^\nu\phi\ V'(|\phi|^2) - \phi\partial^\nu\phi^*\ V'(|\phi|^2) = 0
\end{aligned}$$

where we use the equations of motion in the 3rd equality. Thus the Noether currents are conserved.

8.323 Relativistic Quantum Field Theory I
Spring 2023