Quantum Field Theory I (8.323) Spring 2023
Assignment 1

Feb. 6, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Sec. 2.1 and 2.2
- Weinberg vol 1 Chap. 1

Review of Special Relativity: Lorentz transformations

- We use the notation

$$x^\mu = (x^0, x^i) = (t, x^1, x^2, x^3) = (t, \vec{x}) \ . \tag{1}$$

When used in the argument of a function we often simply write x^μ as x, e.g.

$$\phi(x^\mu) \equiv \phi(x) \ . \tag{2}$$

The four momentum is written as

$$p^\mu = (p^0, p^i) = (E, \vec{p}) \tag{3}$$

and the derivative

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i} \right) = (\partial_t, \nabla) \ . \tag{4}$$

- We use the mostly-plus form of the Minkowski metric, i.e.

$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) = \eta_{\mu\nu} \tag{5}$$

where $\mathrm{diag}(\cdots)$ denotes a diagonal matrix with diagonal entries given by $\cdots$. $\eta_{\mu\nu}$ is the inverse of $\eta^{\mu\nu}$, as

$$\eta_{\mu\lambda}\eta^{\lambda\nu} = \delta_\mu{}^\nu \tag{6}$$

where $\delta_\mu{}^\nu$ is the Kronecker delta symbol.

- Note

$$x_\mu = \eta_{\mu\nu} x^\nu = (-t, \vec{x}), \qquad p_\mu = (-p^0, p^i) = (-E, \vec{p}) \ . \tag{7}$$

- We will also use the notation

$$x^2 \equiv x^\mu x_\mu = -t^2 + \vec{x}^2, \tag{8}$$

$$p^2 \equiv p_\mu p^\mu = -E^2 + \vec{p}^2 \tag{9}$$

and

$$p \cdot x \equiv p_\mu x^\mu = p^\mu x_\mu = -Et + \vec{x} \cdot \vec{p} \ . \tag{10}$$

- A Lorentz transformation acts on x^μ and p^μ as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu \tag{11}$$

where the matrix $\Lambda^\mu{}_\nu$ satisfies the relation

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda \eta^{\rho\lambda} = \eta^{\mu\nu} \tag{12}$$

or in a matrix notation

$$\Lambda \eta \Lambda^t = \eta \tag{13}$$

where the superscript t denotes transpose. We can raise and lower the indices of Λ by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, and equation (13) can also be written as

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\lambda \eta_{\rho\lambda} = \eta_{\mu\nu} \ . \tag{14}$$

- Under a Lorentz transformation (11), a scalar field transforms as

$$\phi(x) \to \phi'(x') = \phi(x) \ ; \tag{15}$$

a vector field transforms as

$$A_\mu(x) \to A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x) \ ; \tag{16}$$

a second rank tensor field transforms as

$$T_{\mu\nu}(x) \to T'_{\mu\nu}(x') = \Lambda_\mu{}^\lambda \Lambda_\nu{}^\rho T_{\lambda\rho}(x) \tag{17}$$

and so on.

- Infinitesimal Lorentz transformations take the form

$$\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \omega_\mu{}^\nu \tag{18}$$

where

$$\omega_{\mu\nu} = -\omega_{\nu\mu}, \qquad \omega_\mu{}^\nu = \eta^{\nu\lambda}\omega_{\mu\lambda} \tag{19}$$

are infinitesimal numbers.

Problem Set 1

1. **Review: Quantum harmonic oscillator in the Heisenberg picture (25 points)**

   Consider the Hamiltonian for a unit mass harmonic oscillator with frequency ω

   $$H = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{x}^2) \ . \tag{20}$$

   In the Heisenberg picture $\hat{p}(t)$ and $\hat{x}(t)$ are dynamical variables which evolve with time. They obey the equal-time commutation relation

   $$[\hat{x}(t), \hat{p}(t)] = i \ . \tag{21}$$

   Here and below we set $\hbar = 1$.

   (a) Obtain the Heisenberg evolution equations for $\hat{x}(t)$ and $\hat{p}(t)$.

   (b) Suppose the initial conditions at $t = 0$ are given by

   $$\hat{x}(0) = \hat{x}, \qquad \hat{p}(0) = \hat{p} \tag{22}$$

   find $\hat{x}(t)$ and $\hat{p}(t)$.

   (c) It is convenient to introduce operators $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ defined by

   $$\hat{x}(t) = \sqrt{\frac{1}{2\omega}}(\hat{a}(t) + \hat{a}^\dagger(t)), \qquad \hat{p}(t) = -i\sqrt{\frac{\omega}{2}}(\hat{a}(t) - \hat{a}^\dagger(t)) \ . \tag{23}$$

   Show that $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ satisfy equal-time commutation relation

   $$[\hat{a}(t), \hat{a}^\dagger(t)] = 1 \ . \tag{24}$$

   (d) Express the Hamiltonian in terms of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

   (e) Obtain the Heisenberg equations for $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

   (f) Suppose the initial conditions at $t = 0$ are given by

   $$\hat{a}(0) = \hat{a}, \qquad \hat{a}^\dagger(0) = \hat{a}^\dagger \tag{25}$$

   find $\hat{a}(t)$ and $\hat{a}^\dagger(t)$.

   (g) Express $\hat{x}(t), \hat{p}(t)$ and the Hamiltonian H in terms of $\hat{a}$ and $\hat{a}^\dagger$.

2. **Review: Lorentz transformations (15 points)**

(a) Prove that the four-dimensional δ-function

$$\delta^{(4)}(p) = \delta(p^0)\delta(p^1)\delta(p^2)\delta(p^3) \tag{26}$$

is Lorentz invariant, i.e

$$\delta^{(4)}(p) = \delta^{(4)}(\tilde{p}) \tag{27}$$

where $\tilde{p}^\mu$ is a Lorentz transformation of p.

(b) Show that

$$\omega_1 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \tag{28}$$

is Lorentz invariant, i.e.

$$= \omega_1' \delta^{(3)}(\vec{k}_1' - \vec{k}_2') \ . \tag{29}$$

$\vec{k}_1$ and $\vec{k}_2$ are respectively the spatial part of four-vectors $k_1^\mu = (\omega_1, \vec{k}_1)$ and $k_2^\mu = (\omega_2, \vec{k}_2)$ which satisfy the on-shell condition

$$k_1^2 = k_2^2 = -m^2 \ . \tag{30}$$

$k_1'^\mu = (\omega_1', \vec{k}_1')$ and $k_2'^\mu = (\omega_2', \vec{k}_2')$ are related to k_1^μ, k_2^μ by a same Lorentz transformation.

(c) For any function $f(k) = f(k^0, k^1, k^2, k^3)$ prove that

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k), \qquad \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} \tag{31}$$

is Lorentz invariant in the sense that

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(k) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} f(\tilde{k}) \tag{32}$$

where $\tilde{k}^\mu = \Lambda^\mu{}_\nu k^\nu$ is a Lorentz transformation of k^μ.

3. **A complex scalar field (20 points)**

Consider the field theory of a complex value scalar field $\phi(x)$ with action

$$S = \int d^4x \left[-\partial_\mu \phi^* \partial^\mu \phi - V(|\phi|^2) \right], \qquad |\phi|^2 = \phi\phi^* \ . \tag{33}$$

One could either consider the real and imaginary parts of ϕ, or ϕ and ϕ^* as independent dynamical variables. The latter is more convenient in most situations.

(a) Check the action (33) is Lorentz invariant (see (15)) and find the equations of motion.

(b) Find the canonical conjugate momenta for ϕ and ϕ^*, and the Hamiltonian H for (33).

(c) The action (33) is invariant under transformation

$$\phi \to e^{i\alpha}\phi, \qquad \phi^* \to e^{-i\alpha}\phi^* \tag{34}$$

for arbitrary constant α. When α is small, i.e. for an infinitesimal transformation, (34) become

$$\delta\phi = i\alpha\phi, \qquad \delta\phi^* = -i\alpha\phi^* \tag{35}$$

Use Noether theorem to find the corresponding conserved current j^μ and conserved charge Q.

(d) Use equations of motion of part (a) to verify directly that j^μ is indeed conserved.

4. **The energy-momentum tensor for the complex scalar field theory (20 points)**

In this problem we work out the energy-momentum tensor of the complex scalar theory (33).

(a) Under a spacetime translation

$$x^\mu \to x'^\mu = x^\mu + a^\mu \tag{36}$$

a scalar field transforms as

$$\phi'(x') = \phi(x) \ . \tag{37}$$

Show that the action (33) is invariant under transformation $\phi(x) \to \phi'(x)$.

(b) Write down the transformation of the scalar fields ϕ and ϕ^* for an infinitesimal translation, and use Noether theorem to find the corresponding conserved currents $T^{\mu\nu}$.

(c) The conserved charge for a time translation

$$H = \int d^3x \, T^{00} \tag{38}$$

should be identified with the total energy of the system, while that for a spatial translation

$$P^i = \int d^3x \, T^{0i} \tag{39}$$

should be identified with the total momentum. Thus $T^{\mu\nu}$ is referred to as the energy-momentum tensor. Write down the explicit expressions for H and P^i. Compare H obtained here with the Hamiltonian of problem 3(b).

(d) Use equations of motion of problem 3(a) to verify directly that $T^{\mu\nu}$ is indeed conserved.

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