Class 10 in-class problems, 18.05, Spring 2022

Concept questions

Concept question 1. Is it a statistic?

You believe that the lifetimes of a certain type of lightbulb follow an exponential distribution with parameter λ. To test this hypothesis you measure the lifetime of 5 bulbs and get data $x_1, \ldots x_5$.

Which of the following are statistics?

(a) *The sample average $\overline{x} = \frac{x_1+x_2+x_3+x_4+x_5}{5}$.*

(b) *The expected value of a sample, namely $1/\lambda$.*

(c) *The difference between $\overline{x}$ and $1/\lambda$.*

Solution: Only (a) is a statistic. Parts (b) and (c) need the value of λ, which is a parameter of the distribution. It cannot be computed from the data. It can only be estimated. So any computation that needs the true value of λ does not produce a statistic.

Board questions

Problem 1. Coins

(a) *A box contains 3 coins. They land heads with, respectively, probability $p = 1/3,\ 1/2,\ 2/3$.*

A coin is taken from the box. The mystery coin is tossed 80 times, resulting in 49 heads and 31 tails.

What is the likelihood of this data for each type of coin? Which coin gives the maximum likelihood?

(b) *Now suppose you found a bent coin. It has an unknown probability p of landing heads. To estimate p you toss it 80 times getting 49 heads. Find the likelihood and log likelihood functions given this data. What is the maximum likelihood estimate for p?*

Solution: **(a)** The data D is 49 heads in 80 tosses.

We have three hypotheses: the coin has probability $p = 1/3$, $p = 1/2$, $p = 2/3$. So the likelihood function $P(D|p)$ takes 3 values:

$$P(D|p=1/3) = \binom{80}{49}\left(\frac{1}{3}\right)^{49}\left(\frac{2}{3}\right)^{31} = 2.07 \cdot 10^{-7}$$

$$P(D|p=1/2) = \binom{80}{49}\left(\frac{1}{2}\right)^{49}\left(\frac{1}{2}\right)^{31} = 0.0118$$

$$P(D|p=2/3) = \binom{80}{49}\left(\frac{2}{3}\right)^{49}\left(\frac{1}{3}\right)^{31} = 0.0545$$

The maximum likelihood is when $p = 2/3$ so this our maximum likelihood estimate is that $p = 2/3$.

(b) Our hypotheses now allow p to be any value between 0 and 1. So our likelihood function is

$$P(D|p) = \binom{80}{49} p^{49}(1-p)^{31}$$

To compute the maximum likelihood over all p, we set the derivative of the log likelihood to 0 and solve for p:

$$l(p) = \ln(P(D|p)) = \ln\left(\binom{80}{49}\right) + 49\ln(p) + 31\ln(1-p)$$
$$\frac{dl(p)}{dp}\,\frac{49}{p} - \frac{31}{1-p} = 0$$
$$\Rightarrow\ p = \frac{49}{80}$$

So our MLE is $\hat{p} = 49/80$.

Problem 2. Continuous likelihood

For continuous likelihood: use the pdf instead of the pmf

Box of light bulbs.

Lifetime of each bulb $\sim$ *exp*(λ), *with unknown parameter* λ.

For multiple independent data points, the likelihood is the product of the individual likelihoods.

(a) *We test 5 light bulbs and find they have lifetimes of 2, 3, 1, 3, 4 years respectively. We assume the tests are independent.*

(i) Find the likelihood and log likelihood functions (as functions of λ*.)*

(ii) What is the maximum likelihood estimate (MLE) for λ*?*

Reminder: An exponential distribution has pdf $f(x|\lambda) = \lambda e^{-\lambda x}$

(b) *Suppose we test 5 bulbs and find they have lifetimes* x_1, x_2, x_3, x_4, x_5 *years respectively. Redo Part (a) using these lifetimes.*

Solution: (a)(i) For a single data value x, the likelihood function is $f(x|\lambda) = \lambda e^{-\lambda x}$. So, given our data, the likelihood functions for the 5 bulbs are

$$f(2|\lambda) = \lambda e^{-2\lambda},\ f(3|\lambda) = \lambda e^{-3\lambda},\ f(1|\lambda) = \lambda e^{-1\lambda},\ f(3|\lambda) = \lambda e^{-3\lambda},\ f(4|\lambda) = \lambda e^{-4\lambda}.$$

We multiply these together to get the joint likelihood

$$L(\lambda) = f(2,3,1,3,4|\lambda) = \lambda e^{-2\lambda}\cdot\lambda e^{-3\lambda}\cdot\lambda e^{-1\lambda}\cdot\lambda e^{-3\lambda}\cdot\lambda e^{-4\lambda} = \lambda^5 e^{-13\lambda}.$$

The log likelihood is $l(\lambda) = 5\ln(\lambda) - 13\lambda$,

(ii) We use calculus to find the maximum likelihood

$$l'(\lambda) = \frac{5}{\lambda} - 13 = 0\ \Rightarrow\ \boxed{\hat{\lambda} = \frac{5}{13}.}$$

The MLE for λ is $\frac{5}{13}$.

(b)(i) For this problem we just replace the explicit numbers 2, 3, 1, 3, 4 by symbols.

$$\begin{aligned} \text{Likelihood } &= L(\lambda) = f(x_1, x_2, x_3, x_4, x_5|\lambda) = \lambda^5 e^{-(x_1+x_2+x_3+x_4+x_5)\lambda} \\ \text{Log likelihood } &= l(\lambda) 5\ln(\lambda) - \left(\sum x_i\right)\lambda. \end{aligned}$$

(ii) The calculus is identical to that in part (a).

$$l'(\lambda) = \frac{5}{\lambda} - \sum x_i = 0 \;\Rightarrow\; \boxed{\hat{\lambda} = \frac{5}{\sum x_i}.}$$

The MLE for λ is $\frac{5}{\sum x_i}$.

Note: This is not surprising: we know the mean of an exponential distribution is $1/\lambda$. The MLE for λ is one over the data mean.

Extra

Cilantro problem *In the Cilantro experiment, assume 55 out of 100 people said Cilantro tastes like soap. Find the maximum likelihood estimate for p, the true proportion of people who feel that way.*

Solution: The likelihood function for p is $L(p) = \binom{100}{55} p^{55}(1-p)^{45}$.

Method 2. Log likelihood

Because the log function turns multiplication into addition it is often convenient to use the log of the likelihood function

$$\text{log likelihood} = \ln(\text{likelihood}) = \ln(P(\text{data} \mid p)).$$

In our example

$$\text{Log likelihood } \; l(p) = \ln\left(\binom{100}{55}\right) + 55\ln(p) + 45\ln(1-p).$$

(Note: The first term is just a messy constant.)

Now we can set the derivative of $l(p)$ to 0 to find the MLE.

$$l'(p) = \frac{55}{p} - \frac{45}{1-p} = 0.$$

This is easy to solve for p. We get $\hat{p} = 0.55$.

Adding a hat is a standard way of indicating an estimate, i.e. $\hat{p}$ is an estimate of the unknown parameter p